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DOCTOR OF PHILOSOPHY THESIS

Study and Performance Evaluation of Coordinate-Convex Policies in Call Admission Control

Settore concorsuale 09/F2 - TELECOMUNICAZIONI

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The progressive development of man
is vitally dependent on invention.
It is the most important product of
his creative brain. Its ultimate purpose
is the complete mastery of mind over
the material world, the harnessing of
the forces of nature to human needs.

Nikola Tesla,
MY INVENTIONS, Electrical Experimenter, 1919.

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Introduction

The Quality of Service (QoS), from network viewpoint, represents the ability of a network element (e.g. an application, host or router) to have some level of assurance that its traffic and service requirements can be satisfied. Telecommunications networks, therefore, have to be managed by several forms of control to maintain a desired level of performance, especially when a limited set of resources like buffers, bandwidth, energy and processing capacity is available [1]. The technology evolution and the users' need have brought to study and implement algorithms and mechanisms able to provide Quality of Service in telecommunication networks. Heuristics often help finding efficient and simple solutions, but it necessary find analytical solutions in order to obtain even more optimised performance [2]. The importance of QoS, and the need of investigating proper optimisation techniques is complicated by the fact that telecommunication networks are currently characterised by a heterogeneity articulated in different ways.

- from the applications point of view: many applications require a specific level of assurance from the network. Examples may be assured database access to retrieve information, tele-medicine, remote control of robots in hazardous environments, financial operations, purchase and delivery, tele-learning, applications for emergencies and security, data acquisition from sensors and sensors control;

- from the technological point of view: several technologies are available at the physical layers (e.g., optical fiber, wireless and satellite systems) to support the network services. Moreover, different protocol solutions at the higher layers are available to carry the information with different functionalities and performance;

- from the user's point of view: users show different responsiveness with respect to the offered services and the tariff structure.

In order to understand the effectiveness of QoS mechanisms on the overall performances it is reported a practical case study. More details are found in [3, 4]. The case study presents an experimental approach to provide a guaranteed QoS over a satellite network based on the Internet Protocol (IP) [5]. The aim is to get a proper environment for data, voice and video transmission to be used for tele-working and distance learning through videoconference. In short, the videoconference service is experimentally provided through the network both applying two bandwidth reservation control mechanisms within the intermediate routers to protect audio flows and not implementing any bandwidth protection scheme. The three alternatives are identified as Full-Control, Light-Control and No-Control. Figure 1 contains a small portion

Bit/rate	Video packet loss percentage			Audio packet loss percentage		
	No-control [%]	Light-control [%]	Full-control [%]	No-control [%]	Light-control [%]	Full-control [%]
128	0.05	0.40	0.05	0	0.10	0.09
256	21	7.88	6.50	31.45	0.40	0.23
384	34.73	22	7.74	38.70	0.08	0
512	46.05	33.10	12.92	50.98	0	0

Figure 1: Objective metrics.

of the results in [4] but sufficient to have an idea of the importance of QoS mechanisms. The case of Full-Control provides the best results in terms of video packet loss and audio packet loss.

One of the most important QoS mechanisms is represented by Call Admission Control. Call Admission Control (CAC) determines when to accept or reject a new connection, flow, or call request (depending on technology used), thus limiting the load that enters a network. This is accomplished by verifying if enough resources are available to satisfy the performance requirements (in terms, e.g., of packet loss, delay, jitter) of an incoming call without penalising the ones already in progress.

A basic model for CAC is the stochastic knapsack in which the objects arrive to and depart from the knapsack at random times. Each objects in the knapsack generates revenue. The optimization problem considered is to accept/block arriving objects as a function of the current system state (i.e., the current number of objects of each class in the knapsack) in order to maximize the long-run average revenue. The stochastic knapsack model can be extended by introducing the concept of feasibility region. This is a (typically bounded) region in the call space, where given QoS requirements in terms, e.g., as said of packet-loss/packet-delay probability, are statistically guaranteed.

This thesis is dedicated to the study of optimality conditions for Call Admission Control (CAC) problems with nonlinearly-constrained feasibility regions and K classes of users. Call admission strategies are restricted to the family of coordinate-convex policies. For two classes of users, different contributions about: general structural properties, number of optimal policies, structural properties depending on the revenue ratio and, robustness to perturbations in the feasibility region boundaries are introduced. Then, the analysis is generalized to the case of $K \geq 2$ classes of users. Compared with previous studies on optimality of the complete-sharing policy, less restrictive conditions are imposed. The theoretical results are exploited for narrowing the set of coordinate-convex admissible policies. Up to our knowledge, the problem of deriving structural properties of the optimal CC policies in the case of general nonlinearly-constrained feasibility regions has received little attention till now.

Part of these results have been published in [6, 7]. Another paper is in preparation [8] and contain most of the results presented in this thesis.

The thesis is organized as follows.

Chapter 1 is dedicated to an introducion of QoS. In Chapter 1.1, QoS issues are addressed for a telecommunication network, highlighting the functionalities necessary to manage the QoS provision. In Chapter 1.2, Call Admission Control Models are introduced. The concepts of stochastick knapsack and feasibility region are explained in the context of CAC.

Chapter 2 is dedicated to the original contribution of the thesis. In Chapter 2.1, the case of 2 classes of users is analyzed: general structural properties, an algorithm to enumerate the set of candidate optimal coordinate-convex policies, structural properties that depend on the revenue ratio associated with the two classes of users are presented. In Chapter 2.2 a generalization to $K \geq 2$ classes of users of the results in Chapter 2.1 is presented. In Chapter 2.3 numerical simulation are detailed. In Chapter 2.4 a conclusive discussion and comparisons with the results derived in [9] for the complete-sharing policy are described.

Chapter 3 is dedicated to a conclusive discussion and comparisons with the results derived in [9] for the complete-sharing policy.

Contents

Chapter 1

Quality of Service in Future Networks

1.1 Introduction to Quality of Service

1.1.1 QoS Definition

There are several definition of Quality of Service. According to ISO 8402, the word quality is defined as "the totality of characteristics of an entity that bear on its ability to satisfy stated and implied needs." ISO 9000 defines quality as the degree to which a set of inherent characteristics fulfils requirements. ITU-T (Recommendation E.800 [10]) and ETSI [11] basically defines QoS as "the collective effect of service performance which determine the degree of satisfaction of a user of the service". As stated in reference [12], IETF considers QoS as the ability to segment traffic or differentiate between traffic types in order for the network to treat certain traffic flows differently from others. QoS encompasses both the service categorization and the overall performance of the network for each category.

Concerning the network viewpoint, QoS is the ability of a network element (e.g. an application, host or router) to have some level of assurance that its traffic and service requirements can be satisfied.
As stated in [1], three types of QoS exist: intrinsic, perceived and assessed [13, 14]. Intrinsic QoS is directly provided by the network itself and, for example, it can be described in terms of objective parameters as, loss and delay. Perceived QoS (P-QoS) is the quality perceived by the users; it heavily depends on the network performance but it is measured by the "average opinion" of the users (Mean Opinion Score, MOS methods are often used to perform the measure). The last term reported concerns assessed QoS. It is referred to the will of a user to keep on using a specific service. It is related to P-QoS but also depends on the pricing mechanism, on the level of assistance of the provider and on other marketing and commercial aspects.
At the moment, most of the QoS provision is offered in terms of intrinsic (objective parameters) QoS by using a Service Level Specification (SLS) which is "a set of parameters and their values which together define the service offered to a traffic" [15]. SLS is a separated technical part of "a negotiated agreement between a customer and the service provider on levels of service characteristics and the associated set of metrics" [13, 16], which is the commonly adopted definition of a Service Level Agreement (SLA). A possible example of SLS is depicted in Figure 1.1 and, derived from the Asynchronous Transfer Mode (ATM) Traffic Contract [17], it is composed of type of traffic, traffic description and conformance testing (packet dimension, application peak and average rate, and, if requested, maximum burst and bucket size); and performance guarantees (packet loss rate, packet transfer delay and packet delay jitter). Moreover, it includes also the feature of Multi-Level Precedence and Pre-emption (MLPP), which is a peculiar characteristic of military voice

switches but, in military environment, is recommendable also for data traffic. It is of topical importance in tactical environment and establishes a level of priority for calls by using 4 bits [18, 19].

Service Level Specification	Range
Connection type	Constant Bit Rate (CBR)
Scope	End-to-end
Connection identification	Sequence of identifiers
Traffic description and conformance testing	Peak rate = 64 Kbps/bucket size for peak rate = 512 bytes/maximum burst size not applicable
Performance guarantees	Packet loss rate = 1%/packet transfer delay = 250 ms/packet delay jitter = 30 ms
Multi Level Precedence and Preemption (MLPP)	Priority

Figure 1.1: Operative example of SLS.

Many applications need QoS. Some of them are: telemedicine, tele-control (remote control of robots in hazardous environments, remote sensors and systems for tele-manipulation), tele-learning, telephony, videoconferences and applications for emergencies and security. Each application, having very different characteristics, deserves a specific degree of service, defined at the application layer. Several standardization bodies have tried to define service categories (also called "QoS classes", to be intended at application layer). ITU-T (in Recommendation Y-1541 [20]) suggests a definition of QoS classes (for the IP world) that is summarized in Figure 1.2.

A further step is to associate objective QoS requirements to QoS traffic classes generically defined above. Concerning the IP environment, the QoS objective metrics mostly used [21] are as follows:

- IPLR - IP Packet Loss Ratio

- IPTD - IP Packet Transfer Delay

QoS class	Characteristics
0	Real-time, jitter sensitive, highly interactive
1	Real-time, jitter sensitive, interactive
2	Transaction data, highly interactive
3	Transaction data, interactive
4	Low loss only (short transactions, bulk data, video streaming)
5	Traditional applications of default IP networks

Figure 1.2: ITU-T Y-1541 QoS classes.

- IPDV - IP Packet Delay Variation (known as Jitter)
- IPER - IP Packet Error Ratio.

Another metric often considered is the skew, which is the average value of the difference of the delays measured by packets belonging to different media, as, for example, voice and video within a videoconference service. In this case, if the skew is large, there is no synchronization between voice and video with the general effect of a bad dubbing. Possible end-to-end performance-metric upper bounds are reported in Figure 1.3, associated with QoS classes in [20].

QoS Class	Characteristics	IPTD	IPDV	IPLR	IPER
0	Real time, jitter sensitive, highly interactive	100 ms	50 ms	1×10^{-3}	1×10^{-4}
1	Real time, jitter sensitive, interactive	400 ms	50 ms	1×10^{-3}	1×10^{-4}
2	Transaction data, highly interactive	100 ms	U	1×10^{-3}	1×10^{-4}
3	Transaction data, interactive	400 ms	U	1×10^{-3}	1×10^{-4}
4	Low loss only (short transactions, bulk data, video streaming)	1 s	U	1×10^{-3}	1×10^{-4}
5	Traditional applications of default IP networks	U	U	U	U

Figure 1.3: IP QoS classes and objective performance-metric upper limits.

1.1.2 QoS Management functions

In order to guarantee specific QoS requirements, QoS management is strongly necessary and QoS management functions are aimed at offering the necessary tools to pursue this objective. A possible classification of the QoS Management functions is presented below from [1]. Others classifications can be found in [22, 23, 24].

Over Provisioning "There is a common misconception that purchasing an oversupply of bandwidth will solve all service-quality challenges. Throwing bandwidth at the problem is sometimes perceived as a simpler solution than QoS management" [25]. This approach ignores not only bandwidth optimization but also possible future trends and requirements of new services. It cannot be classified exactly as a solution; QoS management is strongly necessary and QoS management functions are aimed at offering the necessary tools to get a certain level of quality.

Flow Identification The identification of packets so that they may receive a different treatment within the network is topical to guarantee QoS, ranging from a minimum priority-based service to quality assurance for a specific flow. Different technologies, show different methods to classify packets: Flow Label and Traffic Class in IPv6 [26], ToS and vector "IP source address, IP destination address, Protocol, TCP/UDP source port, TCP/UDP destination port" in IPv4, VPI/VCI in ATM, Label Value in MPLS [27].

Call Admission Control An accurate resource reservation to guarantee that traffic flows receive the correct service is strictly needed. The acceptance/rejection of a new connection is performed subject to a check (that may be statistical) about the availability of network resources in consequence of specific requirements. After that, if enough resources are available, they are reserved. The next section, as well as, the rest of the thesis, will be devoted to CAC.

Traffic Control (Shaping) Shaping policies limit flows to their committed rates (e.g. the flows need to be conform with their traffic descriptors). They are very important to guarantee performance requirements. If flows/connections exceed their bandwidth consumption specifications, the network, which has dimensioned resources in strict dependence on the declarations, cannot guarantee any specified QoS requirement. Two common methods are used in literature to shape traffic: leaky bucket and token bucket [28].

Scheduling Packet scheduling specifies the service policy of a queue within a node (e.g. an IP router, an ATM switch). In practice, scheduling decides the order that is used to pick the packets out of the queue and to transmit them over the channel. It has a strong impact on different QoS parameters as delay, jitter and loss. The main problem arises from the impossibility of assigning the committed bandwidth to a specific flow at each time instant. The only scheduler that allocates the overall outgoing bandwidth to all users in progress in strict proportion with the bandwidth allocated, for each time instant, is the Generalized Processor Sharing (GPS). Real systems have to use alternative schemes aimed at performing as closest as possible to GPS. A clear and complete revision of the most interesting schedulers is reported in [29].

Queue Management Scheduling is often linked to queue (buffer) management schemes. They are used, for example, to establish the dropping strategy when the buffer is full. Possible strategies include: tail drop (which discharges the last arrived packets), front drop (which eliminates the first packet in the queue), random drop (which discharges a randomly selected packet within the queue) or dynamic schemes.

Flow Control The flow control is the process of managing the rate of data transmission between two nodes to prevent a fast sender from outrunning a slow receiver. It provides a mechanism for the receiver to control the transmission speed, so that the receiving node is not overwhelmed with data from transmitting node. In some cases, the bit rate entering the network may be ruled according to a congestion notification (ECN - Explicit Congestion Notification). Some protocols (e.g. TCP) consider packet loss as a congestion indication. Generally, flow control is implemented end-to-end at the transport layer (but some mechanisms are implemented at the application layer); even if it may help avoiding network saturation, it cannot guarantee, if used alone, a specific QoS requirement.

QoS Routing Packet routing decisions are often taken with little or no awareness of network state and resource availability. This is not compatible with QoS provision. The routing problem cannot be considered as separated from the other controls, but it should be integrated with them. Moreover, the particular nature of QoS-oriented traffic-engineered networks, where the dynamic structure of statistical multiplexing must be combined with resource allocation techniques, allows getting suggestions from the vast previous experience in routing from both circuit- and packet-switched networks.

The intervention timescale of the mentioned functions is different. Figure 1.4 shows a possible time mapping for them.

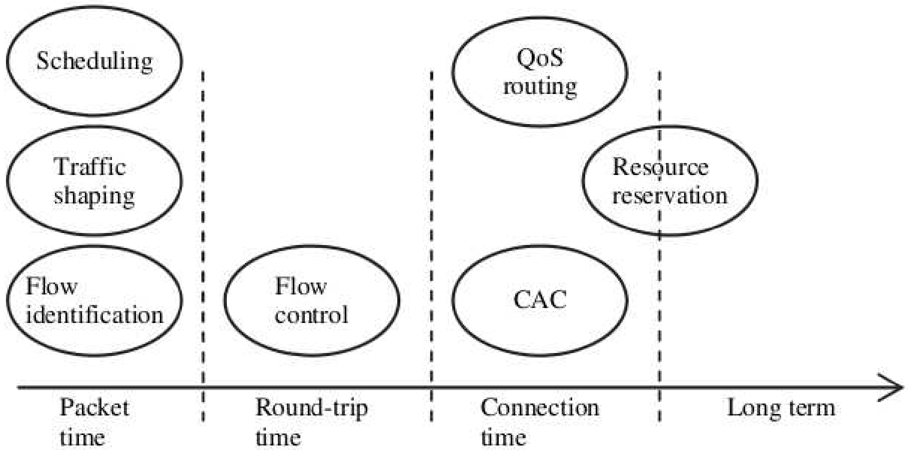


Figure 1.4: QoS management functions versus time.

1.2 Call Admission Control Models

Call Admission Control decides whether a new connection request may be accepted or not. It is a powerful tool to guarantee quality because it allows limiting the load entering the network and verifying if enough resources are available to satisfy the requested performance requirements of a new call without penalizing the connections already in progress. A connection provides traffic descriptors (e.g. the set of traffic parameters of an ATM source) and QoS requirements (i.e. a specific SLS). The network evaluates if there are sufficient resources in terms of bandwidth and buffer to match the request and decides whether to accept or reject the connection.

1.2.1 The Risk of No Control

The aim of this section (from [1]) is to show the effect of CAC. To get the results shown in the following, each single network node is modeled as a battery of buffers that receive the traffic entering the node. An ideal generalized processor sharing (GPS) scheduler is supposed. Most of the results reported concern a single node. The following three data traffics are used. Traffic model and buffer dimension are the same as used in the traffic aggregation analysis. There are three traffic classes, identified as 1, 2 and 3. Class 1 imposes a constraint only on PLR of 10^{-14}. Class 2, additionally to packet loss 10^{-2} , puts a constraint also on transfer delay, set to 10 ms. Class 3 constraints the delay jitter below 5 ms, as well as loss and delay, set, respectively, to 10^{-2} and 10 ms. Traffic of each class enters a separate buffer.

The results show what happens to the performance guarantees if no CAC is implemented, if the requests of new connections to enter the network are not limited. The reference "0%" in the Figures 1.5, 1.6 identifies the situation where traffic entering the network is blocked by CAC and, consequently, the bandwidth has been properly dimensioned and the quality guaranteed. The vertical value in the figures identifies the percentage of traffic above the acceptance threshold (i.e. the percentage of connections that would have been dropped in case of CAC) up to 50% of unbalance. The number of connections ranges from 20 to 300 for Class 1 tests and from 20 to 120 for Classes 2 and 3 tests. Figure 1.5 reports the values of the measured PLR for "Traffic Class 1". Figure 1.6, instead, show the tests for "Traffic Class 2", focusing on 0–30% of non-controlled traffic and reporting the measured PTD.

To conclude: small variations of the number of accepted calls with respect to the reference value (that guarantees the required QoS) have a heavy effect on the traffic flow performance. The reported values allow measuring the performance decrease. The conclusion is that no guarantee is possible if no CAC is implemented.

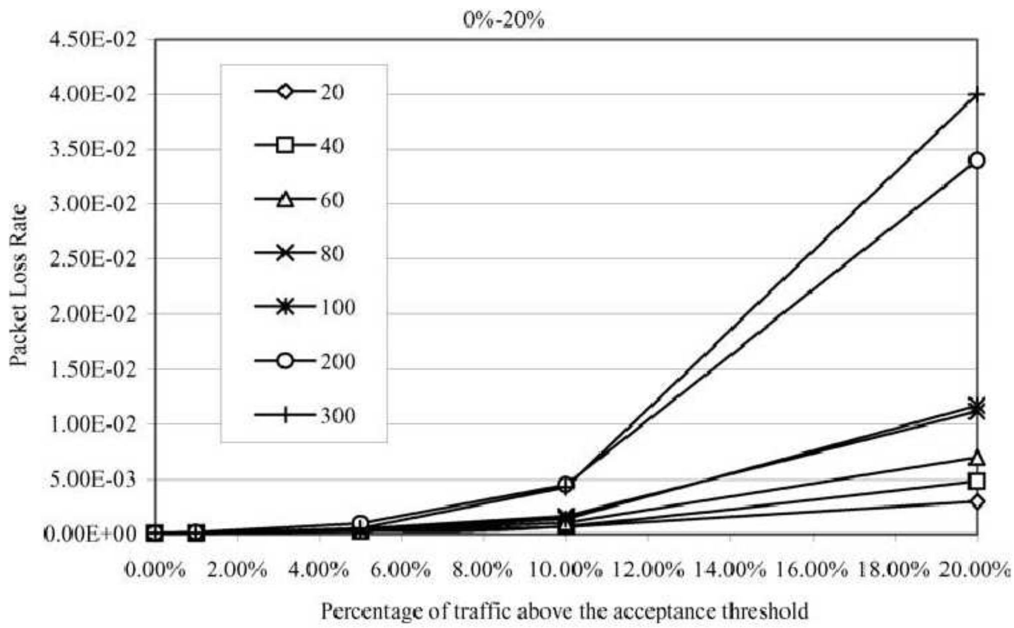


Figure 1.5: QoS management functions versus time.

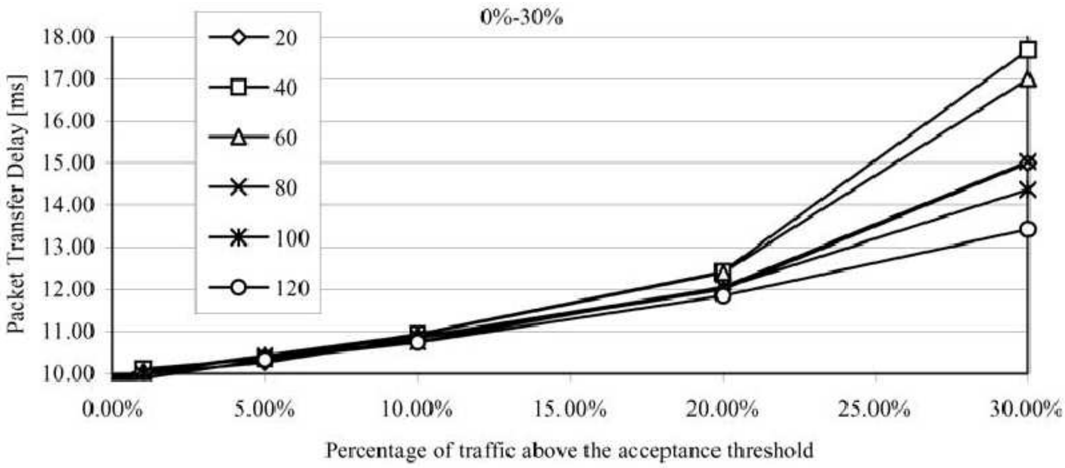


Figure 1.6: QoS management functions versus time.

1.2.2 Stochastic knapsack model

As stated in [30], CAC has been extensively studied in wireline networks as an essential tool for congestion control and QoS provisioning. Different aspects of CAC design and performance analysis, particularly in the context of broadband integrated service digital network (B-ISDN) based on asynchronous transfer mode (ATM) technology, have been investigated in [31]. Nowadays, the problem of CAC is interesting also in wireless networks. CAC schemes can be classified based on various design options. Each design option has its own advantages and disadvantages. In [30] the authors, classify various CAC schemes based on several parameters like centralization, information scale, service dimension, optimization, decision time, information type, information granularity and considered link. A similar characterization (for IP networks), is proposed by [32] in which the main contribution is to summarize an ontology that collects the major characteristics of CAC schemes.
As done in the past by several researchers in this field, we are interested in CAC schemes optimization, in which an object function, dependent on the policy of the CAC, must be maximized.

As reported in [33], a great variety of practical problems, including CAC, can be represented by a set of objects, each having an associated reward value and a volume value, from which a subset has to be selected such that the total reward value is maximized and the total volume does not exceed some prefixed bound. These problems are generally called knapsack problems, in particular *stochastic knapsack problem*. In the stochastic knapsack [34], the objects arrive to and depart from the knapsack at random times. It is assumed that several classes of objects arrive according to independent birth processes. Interarrival times from any given class are assumed to be exponentially distributed with mean depending on the current number of objects of that class in the knapsack. An arriving object bypasses the knapsack if insufficient volume is present. Each objects in the knapsack generates revenue. The optimization problem considered is to accept/block arriving objects as a function of the current system state (i.e., the current number of objects of each class in the knapsack) in order to maximize the long-run average revenue.

To understand how the stochastic knapsack can be model CAC schemes, we introduce the ATM multiplexer model form Ross in [35]. As done in [35], we maintain his notation derived form ATM world. We first give some ATM terminology. When a source wants to transmits information, it requests establishment of a virtual channel (VC). Once a source has established a VC, it generates a stream of cells, each consisting of 53 bytes. A typical cell stream generated by an established VC consists of silent periods, during which no cells are generated, and activity period, during which cells are generated at the peak rate. A ATM multiplexer is a buffer and a high-speed link; the buffer receives the cell generated by established VCs and transmits these cells, one after another, onto the high-speed link. Assume that VCs belong to a finite set of services (or traffic classes). During period when the aggregate cell arrival rate exceeds the link capacity, the multiplexer can significantly delay or even loss cells. To guarantee that all established VCs meet their QoS requirements, the multiplexer may have to deny certain VC establishment requests, thus the need for an admission policy.

A possible admission policy is the Admission Based on Peak Rates. Let C denote the transmission capacity of the high-speed link, K denote the number of services, and $b_1, \ldots, b_K$ denote the peak rates for the K services. The VC profile is $(n_1, \ldots, n_K)$, where n_k is the number of class-k VCs in

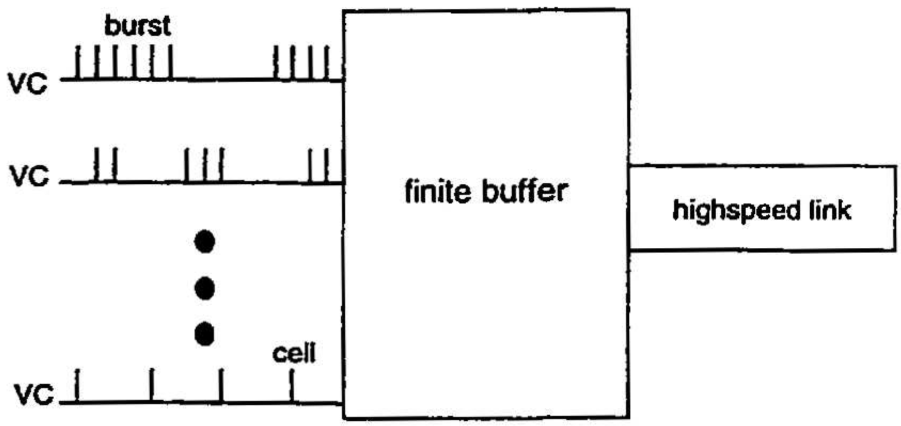


Figure 1.7: An ATM multiplexer [35].

progress. Peak rate admission a new service-k VC if and only if

$$b_k + \sum_{l=1}^{K} b_l n_l \leq C \tag{1.1}$$

This condition ensures that cells experience negligible delay and no loss at the buffer; consequentially, the QoS requirements are met. An ATM multiplexer with peak-rate admission is, with regard to VC dynamics, a loss system and can be modeled as stochastic knapsack.

There are other CAC schemes that can be modeled as stochastic knapsack. Please refer to [31] for an overview.

In general, finding optimal policies for the stochastic knapsack (that is, policies that maximize the objective function) is a difficult combinatorial optimization problem [35, Chapter 4]. The knowledge of structural properties of the optimal policies is useful to simplify its solution or at least to find good suboptimal admission strategies. For instance, for two classes of users and an objective given by a weighted sum of per-class average revenues, structural properties were derived in [34] for the optimal policies belonging to the family of Coordinate-Convex policies (CC policies). Such policies restrict the call state $(n_1, \ldots, n_K)$ of the CAC system to suitable subsets of $\{(n_1, \ldots, n_K) \in \mathbb{N}_0^K : \sum_{k \in \mathcal{K}} n_k b_k \leq C\}$, where each n_k represents the number of calls of the k-th class accepted by the system and currently in progress. CC policies form a large family of CAC policies characterized by a relatively simple structure and interesting properties, such as their product-form steady-state distribution [35, Chapter 4] and bounds on the per-class blocking probabilities [36]. When the service rates and resource requirements do not depend on the customer's classes (single service), the optimal CAC policy is not CC and is called Trunk Reservation ([37, 38]) (TR). In [39] and [40], recursive formulas were derived to evaluate the performance TR and an iterative search algorithm to find optimal policies is described. To this end, the authors exploited the algorithm to find coordinate-optimal threshold policies (a particular kind of CC policies) in multiservice systems (i.e., systems where different classes may have different and heterogeneous resource requirements and mean service times). Prior works on this topic are due to Foschini et.al. [41, 42].

1.2.3 Feasibility Region

An important extension of CAC schemes is the concept of *feasibility region*. The feasibility region is a (typically bounded) region Ω_{FR} in the call/connections space, where given QoS requirements in terms, e.g., of packet-loss/packet-delay probability, are statistically guaranteed. From our knowledge, this separation principle was firstly introduced by [43] and clearly explained in [35] and [44] about ATM multiplexer. Indeed, from [44]:
"Among the various approaches, some of them impose a precise structure to allocate resources. Traffic is typically subdivided into classes, which are

homogeneous in terms of performance requirements and/or statistical characteristics of the traffic sources, and bandwidth is allocated accordingly, thereby restricting the statistical multiplexing only within each class (service separation). This further eases the decomposition of a very complex overall control task, which is generically characterized by very different timescales and requirements, according to the level where the system dynamics is considered (e.g. packet and call level), into smaller and possibly independent problems."

With the feasibility region packet levels and call levels are decoupled, allowing to study the two problems separately. At call level, in fact, it is possible to find a sub-area within the feasibility region (in practice partitioning the feasibility region), in order to satisfy others QoS requirements at call levels (e.g. minimizing call blocking probability).

Moreover, in many cases (e.g. in wireless networks [9], or in ATM network using statistical multiplexing with service separation [35]) the boundary of the feasibility region is *nonlinear*.

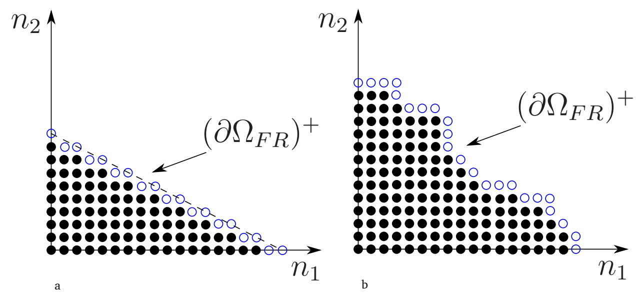


Figure 1.8: The upper boundary $(\partial\Omega_{FR})^+$ of a feasibility region Ω_{FR} with 2 classes of users, in the case of (a) a linearly-constrained Ω_{FR} (stochastic knapsack) and (b) a nonlinearly-constrained Ω_{FR}.

A feasibility region with a nonlinear constraint can be modeled as a modified version of stochastic knapsack in which its size is variable and depends on the number of calls of each class k ($k = 1, \dots, K$) in it. As remarked in [35, pp. 139-140], often the nonlinear part of the boundary is difficult to describe, both analytically and by simulations. In such a situation, it is worth investigating structural properties of the optimal policies and their robustness with respect to perturbations of the feasibility region.

Exploring these issues for the family of CC policies is the aim of this thesis, in which we further develop the approach proposed in [6, 7]. Up to our knowledge, till now the problem of deriving structural properties of the optimal CC policies in the case of general nonlinearly-constrained feasibility regions has received little attention. Some exceptions are [45, 9] and our previous works [6, 7].

Chapter 2

Coordinate-Convex Policies

2.1 The Case of 2 Classes of Users

2.1.1 Problem formulation

The CAC system in [34] is described by a 2-dimensional vector $\mathbf{n}$, whose component n_k, $k = 1, 2$, represents the number of connections of class k that have been accepted by the system and are currently in progress. For each class k, inter-arrival times are exponentially distributed with mean value $1/\lambda_k(n_k)$ and accepted connections holding times are independent and identically distributed (i.i.d.) with mean value $1/\mu_k$. The CAC system accepts or rejects a connection request according to a CC *policy*, whose definition [35, p. 116] is recalled in Definition 2.1.

Definition 2.1. *A nonempty set* $\Omega \subseteq \Omega_{FR} \subsetneq \mathbb{N}_0^2$ *is called CC if and only if for each* $\mathbf{n} \in \Omega$ *with* $n_k > 0$ *one has* $\mathbf{n} - \mathbf{e}_k \in \Omega$, $\forall k = 1, 2$, *where* $\mathbf{e}_k$ *is a 2-dimensional vector whose* k*-th component is* 1 *and the other one is* 0. *The CC policy associated with a CC set* Ω *admits an arriving request of connection if and only if after admittance the state process remains in* Ω.

As there is a one-to-one correspondence between CC sets and CC policies, from now on we use the symbol Ω to denote both a CC set and a CC policy.

We consider the following optimization problem associated to the CAC system:

$$\text{maximize } J(\Omega) = \sum_{\mathbf{n}\in\boldsymbol{\Omega}} (\mathbf{n}\cdot\mathbf{r})P_\Omega(\mathbf{n})\,, \tag{2.1}$$

$$\text{s.t. } \Omega \in \mathcal{P}(\Omega_{FR})\,, \tag{2.2}$$

where $\mathcal{P}(\Omega_{FR})$ is the set of CC subsets of Ω_{FR}, $\mathbf{r} := (r_1, r_2)$ is a 2-dimensional vector whose component r_k represents the instantaneous revenue generated by any accepted connection of class k that is still in progress, and $P_\Omega(\mathbf{n})$ is the steady-state probability that the CAC system is in state $\mathbf{n}$. As Ω is CC, $P_\Omega(\mathbf{n})$ takes on the product-form expression

$$P_\Omega(\mathbf{n}) = \frac{\prod_{k=1}^{2} q_k(n_k)}{\sum_{\mathbf{n}\in\Omega}\prod_{k=1}^{2} q_k(n_k)}, \tag{2.3}$$

where

$$q_k(n_k) := \frac{\prod_{j=0}^{n_k-1}\lambda_k(j)}{n_k!\mu_k^{n_k}}\,. \tag{2.4}$$

In the case of linearly-constrained feasibility region Ω_{FR} ($\{(n_1,\ldots,n_K) \in \mathbb{N}_0^K : \sum_{k\in\mathcal{K}} n_k b_k \le C\}$), [34], as we said, derived sufficient conditions under which the CC policies maximizing the objective (2.1) are of threshold type (defined in Definition 2.19). Such conditions depend on the value assumed by the revenue ratio $R := r_2/r_1$.

For a generic, not necessarily linearly-constrained Ω_{FR}, Proposition 2.2 (see also [46] for a similar approach) states that, in the case of homogeneous Poisson arrivals with rate λ_k for each class k, if $r_1 = \frac{\mu_1}{\sum_{j=1}^{2}\lambda_j}$ and $r_2 = \frac{\mu_2}{\sum_{j=1}^{2}\lambda_j}$, maximizing the objective (2.1) over the set of CC policies $\Omega \subseteq \Omega_{FR}$

is equivalent to minimizing $J'(\Omega)$ in (2.5) over the same set. Proposition 2.2 will be exploited in Section 2.3 to choose the values of some parameters in the numerical simulations. $J'(\Omega)$ is a weighted sum of per-class blocking probabilities. The action of minimizing $J'(\Omega)$ is called *Erlang scheme* [44].

$$J'(\Omega) := \sum_{k=1}^{2} \frac{\lambda_k}{\sum_{j=1}^{2} \lambda_j} \cdot \beta_k(\Omega)\,, \tag{2.5}$$

where $\beta_k(\Omega)$ is the blocking probability for class k (i.e., the probability that an incoming connection request from class k is not accepted by the CC policy Ω).

Proposition 2.2. *For $k = 1, 2$, let the arrivals from class k be homogeneous Poisson with rate λ_k. Set $r_1 = \frac{\mu_1}{\sum_{j=1}^{2} \lambda_j}$ and $r_2 = \frac{\mu_2}{\sum_{j=1}^{2} \lambda_j}$. Then*

$$\mathrm{argmax}_{\Omega \in \mathcal{P}(\Omega_{FR})} J(\Omega) = \mathrm{argmin}_{\Omega \in \mathcal{P}(\Omega_{FR})} J'(\Omega)\,.$$

Proof of Proposition 2.2. Set

- $\bar{n}_k$: average number of class k users;
- r_k: revenue per unit time generated by a class k object;
- $r_k \bar{n}_k$: average revenue per unit time generated by class k users.

Then

$$\begin{aligned} J(\Omega) &= \sum_{\mathbf{n} \in \Omega} \sum_{k=1}^{2} n_k r_k \, P_\Omega(\boldsymbol{n}) \\ &= \sum_{k=1}^{2} r_k \sum_{\mathbf{n} \in \Omega} n_k P_\Omega(\mathbf{n}) \\ &= \sum_{k=1}^{2} r_k \bar{n}_k \,. \end{aligned} \tag{2.6}$$

Set

- L_k: throughput of class k;
- $\frac{1}{\mu_k}$: mean service time for class k objects.

By Little's theorem [47] one has

$$\bar{n}_k = L_k \frac{1}{\mu_k} \tag{2.7}$$

and (2.6) gives

$$J(\Omega) = \sum_{k=1}^{2} r_k L_k \frac{1}{\mu_k} \,. \tag{2.8}$$

So, in general, $J(\Omega)$ is a weighted sum (with weights r_k/μ_k) of the throughputs associated with the 2 classes.

As the arrivals for each class are homogeneous Poisson, it follows from [35, p. 20] that the following relation between throughput and blocking probability holds:

$$L_k = \lambda_k (1 - \beta_k(\Omega)) \,. \tag{2.9}$$

Hence,

$$\begin{aligned} \sum_{k=1}^{2} r_k \frac{1}{\mu_k} L_k &= \sum_{k=1}^{2} r_k \frac{1}{\mu_k} \lambda_k (1 - \beta_k(\Omega)) \\ &= \sum_{k=1}^{2} r_k \rho_k - \sum_{k=1}^{2} r_k \rho_k \beta_k(\Omega) \,. \end{aligned} \tag{2.10}$$

Finally, setting $r_1 = \frac{\mu_1}{\sum_{j=1}^{2} \lambda_j}$ and $r_2 = \frac{\mu_2}{\sum_{j=1}^{2} \lambda_j}$, one gets

$$\begin{aligned} \operatorname{argmax}_\Omega J(\Omega) &= \operatorname{argmax}_\Omega \left(\sum_{k=1}^{2} r_k \rho_k - \sum_{k=1}^{2} r_k \rho_k \beta_k(\Omega) \right) \\ &= \operatorname{argmin}_\Omega \sum_{k=1}^{2} r_k \rho_k \beta_k(\Omega) \\ &= \operatorname{argmin}_\Omega J'(\Omega) \,. \end{aligned} \tag{2.11}$$

■

In our analysis, we consider the general case in which the feasibility region Ω_{FR} may have a nonlinear upper boundary, denoted by $(\partial\Omega_{FR})^+$ (see Figure 1.8(b)). Similarly, we denote by $(\partial\Omega)^+$ the upper boundary of the CC set Ω. Also the set Ω_{FR} is assumed to be CC, as it often happens for feasibility regions defined in terms of quality-of-service constraints [48, Proposition 6.3]. Let us recall from [34]:

Definition 2.3. *The tuple $(\alpha, \beta) \in \Omega_{FR} \setminus \Omega$ is a type-1 corner point for Ω if and only if $\beta \geq 1$, $(\alpha, \beta - 1) \in \Omega$, and either $\alpha = 0$ or $(\alpha - 1, \beta) \in \Omega$; the tuple $(\alpha, \beta) \in \Omega_{FR} \setminus \Omega$ is a type-2 corner point for Ω if and only if $\alpha \geq 1$, $(\alpha - 1, \beta) \in \Omega$, and either $\beta = 0$ or $(\alpha, \beta - 1) \in \Omega$.*

We shall use the term "corner point" to refer both to a type-1 and to a type-2 corner point. No two corner points can be on the same vertical or horizontal line for the coordinate-convexity of Ω.

2.1.2 General structural properties

We recall from [34] that the definition of the objective $J(\cdot)$ in (2.1) can be extended consistently to all (not necessarily CC) sets $S \subseteq \Omega_{FR}$ in the following way:

$$J(S) := \frac{H(S)}{G(S)} \tag{2.12}$$

with

$$H(S) := \sum_{\mathbf{n} \in S} (\mathbf{n} \cdot \mathbf{r}) \prod_{k=1}^{2} q_k(n_k)\,, \tag{2.13}$$

$$G(S) := \sum_{\mathbf{n} \in S} \prod_{k=1}^{2} q_k(n_k)\,. \tag{2.14}$$

For a rectangular region $S := \{a, a+1, \ldots, b\} \times \{c, c+1, \ldots, d\}$, from (2.4), (2.13), and (2.14) follows that

$$J(S) = r_1 x_1(a, b) + r_2 x_2(c, d)\,. \tag{2.15}$$

Loosely speaking, the next Proposition 2.4 states the following. Given a CC policy Ω, if for some type-2 corner point one can find two rectangular regions S^+ and S^- (of the below-defined shape) as in Figure 2.1(b), then Ω is suboptimal, since it can be improved either by adding S^+ or by removing S^-. This is a nontrivial result, since for the objective (2.1) and any two CC sets $\Omega_1, \Omega_2 \subseteq \Omega_{FR}$, the relationship $\Omega_1 \subseteq \Omega_2$ does not imply $J(\Omega_1) \leq J(\Omega_2)$. Note that, although Proposition 2.4 looks similar to [34, Lemma 2], it states a different concept. While [34, Lemma 2] states a property satisfied by any optimal CC policy, Proposition 2.4 provides a constructive way to improve certain suboptimal c.c policies.

Proposition 2.4. *Let* (α, β) *be a type-*2 *corner point for* Ω *and suppose that there exist* $n, m, p \in \mathbb{N}_0$ *such that* $S^- := \{(\alpha - 1 - j, \beta + i) : j = 0, \ldots, n,\ i = 0, \ldots, p\} \subset \Omega$, *is* IR_Ω, *and* $S^+ := \{(\alpha + s, \beta + i) : s = 0, \ldots, m,\ i = 0, \ldots, p\} \subset \Omega_{FR}$, *is* IA_Ω. *Then, at least one of the following inequalities holds: (i)* $J(\Omega \cup S^+) > J(\Omega)$*; (ii)* $J(\Omega \setminus S^-) > J(\Omega)$.

Proof of Proposition 2.4. Suppose that neither $J(\Omega \cup S^+) > J(\Omega)$ nor $J(\Omega \setminus S^-) > J(\Omega)$ holds. Then

$$J(\Omega \cup S^+) = \frac{H(\Omega) + H(S^+)}{G(\Omega) + G(S^+)} \leq J(\Omega) = \frac{H(\Omega)}{G(\Omega)},$$

which implies $J(S^+) = H(S^+)/G(S^+) \leq H(\Omega)/G(\Omega) = J(\Omega)$. Similarly, one gets $J(S^-) \geq J(\Omega)$, so $J(S^-) \geq J(S^+)$. On the other hand, computing $J(S^-)$ and $J(S^+)$ by formula (2.15) one has $J(S^-) = r_1 x_1(\alpha - 1 - n, \alpha - 1) + r_2 x_2(\beta, \beta + p)$, $J(S^+) = r_1 x_1(\alpha, \alpha + m) + r_2 x_2(\beta, \beta + p)$, thus $J(S^-) < J(S^+)$, but this is a contradiction. So, at least one between cases (i) and (ii) holds. ■

Proposition 2.5 states a similar concept for type-1 corner points and is proved by reversing the roles of the two classes.

Proposition 2.5. *Let (α, β) be a type-1 corner point for Ω and suppose that there exist $n, m, p \in \mathbb{N}_0$ such that $S^- := \{(\alpha+i, \beta-1-j) : i = 0, \dots, p,\ j = 0, \dots, n\} \subset \Omega$, is IA_Ω, and $S^+ := \{(\alpha+i, \beta+s) : i = 0, \dots, p,\ s = 0, \dots, m\} \subset \Omega_{FR}$, is IA_Ω. Then, at least one of the following inequalities holds: (i) $J(\Omega \setminus S^-) > J(\Omega)$; (ii) $J(\Omega \cup S^+) > J(\Omega)$.*

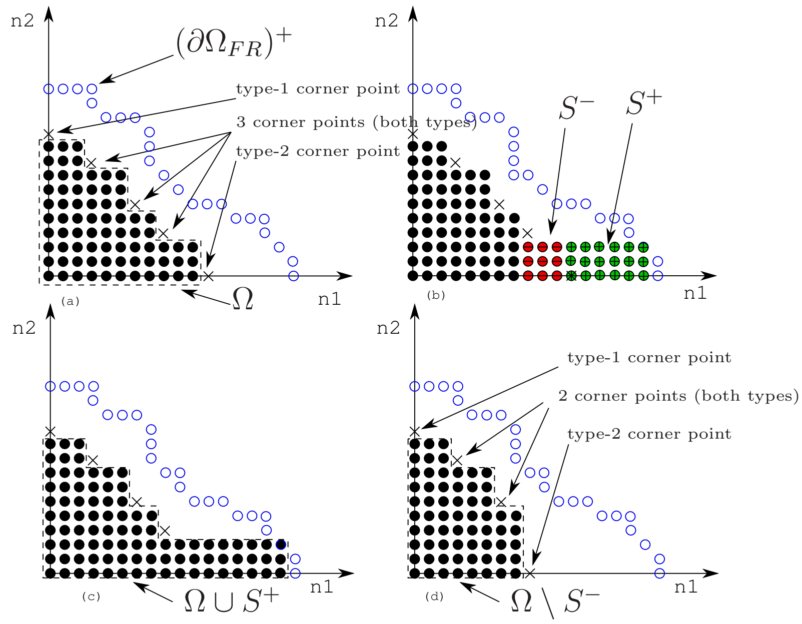


Figure 2.1: The sets Ω, S^+ and S^- in Proposition 2.4.

Proposition 2.6. *Let the assumptions of Proposition 2.4 hold for some p, and $m = n = 0$.*

(i) Let $\bar{m} \geq 0$ be the maximum value of m for which $S^+(m)$ is IA_Ω. If $J(\Omega \cup S^+(0)) > J(\Omega)$, then $J(\Omega \cup S^+(m))$ is an increasing function of m for $m \in \{0, \dots, \bar{m}\}$.

(ii) Let $\bar{n} \geq 0$ be the maximum value of n for which $S^-(n)$ is IR_Ω. If $J(\Omega \setminus S^-(0)) > J(\Omega)$, then $J(\Omega \setminus S^-(n))$ is an increasing function of n for $n \in \{0, \dots, \bar{n}\}$.

Proof of Proposition 2.6. Let us prove (i), supposing that the assumptions of Proposition 2.4 hold for some p and $m = n = 0$. The other case can be proved in a similar way. Let $m \in \{0, \dots, \bar{m}\}$. Then we have

$$\begin{aligned} & J(\Omega \cup S^+(m)) \\ = & \tfrac{H(\Omega)+H(S^+(m))}{G(\Omega)+G(S^+(m))} \\ = & \tfrac{G(\Omega)}{G(\Omega)+G(S^+(m))}\tfrac{H(\Omega)}{G(\Omega)} + \tfrac{G(S^+(m))}{G(\Omega)+G(S^+(m))}\tfrac{H(S^+(m))}{G(S^+(m))} \\ = & \gamma(m)J(\Omega) + (1-\gamma(m))J(S^+(m)) \,, \end{aligned} \tag{2.16}$$

where $\gamma(m) := \frac{G(\Omega)}{G(\Omega)+G(S^+(m))}$. Note that $\gamma(m)$ is a decreasing function of m. Moreover, by (2.15) we get

$$J(S^+(m)) = r_1 x_1(\alpha, \alpha + m) + r_2 x_2(\beta, \beta + p)$$

where $x_1(\alpha, \alpha + m)$ is a nondecreasing function of m (as it can be checked by applying the definition (2.27) of $x_i(\cdot, \cdot)$), so $J(S^+(m))$ is a nondecreasing function of m, too. Let $m_1, m_2 \in \{0, \dots, \bar{m}\}$ and $m_1 < m_2$. Then

$$\begin{aligned} & J(\Omega \cup S^+(m_2)) \\ = & \gamma(m_2)J(\Omega) + (1-\gamma(m_2))J(S^+(m_2)) \\ \geq & \gamma(m_2)J(\Omega) + (1-\gamma(m_2))J(S^+(m_1)) \\ > & \gamma(m_1)J(\Omega) + (1-\gamma(m_1))J(S^+(m_1)) \\ = & J(\Omega \cup S^+(m_1)) \,, \end{aligned} \tag{2.17}$$

which proves that $J(\Omega \cup S^+(m))$ is an increasing function of m for $m \in \{0, \dots, \bar{m}\}$.

■

Propositions 2.4 and 2.6 suggest the following greedy algorithm to improve a given CC policy. For any CC policy Ω, we denote by $I_2(\Omega_j)$ the

1 Choose an initial CC policy Ω_1;
2 **while** $I_2(\Omega_j) \neq \emptyset$ **do**
3 let $\Omega_{j+1} \in \operatorname{argmax}\{\Omega_j \cup S^+(\bar{m}), \Omega_j \setminus S^-(\bar{n}) : (\alpha, \beta) \in I_2(\Omega_j)\}$;
4 **end**
5 return the current policy Ω_j.

Algorithm 1: Algorithm to improve CC policy.

associated set of type-2 corner points for which the assumptions of Proposition 2.4 hold for some p, and $m = n = 0$.

In Step 2 of Algorithm 1, one selects among the type-2 corner points of the current policy Ω_j for which Proposition 2.4 can be applied, the one that guarantees the best local improvement of Ω_j. Such an improvement may be obtained either by adding the set $S^+(\bar{m})$ to Ω_j, or removing the set $S^-(\bar{n})$.

Note that Algorithm 1 terminates after a finite number of iterations, with a CC policy that is not necessarily optimal. However, at each step the policy that guarantees the best local improvement of the Objective (2.1) is selected. The number of iterations is finite for at least one of the following two reasons:

- when $I_2(\Omega_j) \neq \emptyset$, an application of Proposition 2.4 implies $J(\Omega_{j+1}) > J(\Omega)$, and $\mathcal{P}(\Omega_{FR})$ has finite cardinality, so each set Ω_j is visited at most once by the algorithm;

- it follows easily by the definitions of $\bar{m}$ and $\bar{n}$ that, when $I_2(\Omega_j) \neq \emptyset$, one has $|I_2(\Omega_{j+1})| \leq |I_2(\Omega_j)| - 1$. Recall that no two corner points can have a coordinate in common. This implies that, for any CC policy Ω, one has $|I_2(\Omega)| \leq \max\{n_{1,\max}^{FR}, n_{2,\max}^{FR}\} + 1$. So, $|I_2(\Omega_0)| \leq \max\{n_{1,\max}^{FR}, n_{2,\max}^{FR}\} + 1$, and the algorithm terminates after at most $|I_2(\Omega_0)|$ iterations.

Finally, in order to obtain $I_2(\Omega_{j+1})$ from $I_2(\Omega_j)$, only one corner point is removed and at most one coordinate of another corner point is changed. Moreover, let $I_2^-(\Omega_j) \subset I_2(\Omega_j)$ be the set of the type-2 corner points (α, β) of Ω_j for which condition (ii) of Proposition 2.4 is verified, and that are not selected at step j of Algorithm 1. Then, since $J(\Omega_{j+1}) > J(\Omega_j)$, one can see by the proof of Proposition 2.4 that $I_2^-(\Omega_j) \subseteq I_2(\Omega_{j+1})$. So, this property is useful in some cases to check if a corner point of Ω_{j+1} belongs to the set $I_2(\Omega_{j+1})$ of Algorithm 1.

Let Ω^o denote a generic optimal CC policy (or its associated CC set). The next Proposition 2.7 states that Ω^o has a nonempty intersection with the upper boundary $(\partial\Omega_{FR})^+$ of the feasibility region. Note that this is a nontrivial result, as for any two CC sets $\Omega_1, \Omega_2 \subseteq \Omega_{FR}$, in general $\Omega_1 \subseteq \Omega_2$ does not imply $J(\Omega_1) \leq J(\Omega_2)$.

Proposition 2.7. *Ω^o has a nonempty intersection with $(\partial\Omega_{FR})^+$.*

Proof of Proposition 2.7. Let us consider any CC set Ω such that $\Omega \cap (\partial\Omega_{FR})^+ = \emptyset$. We show that, by a repeated application of Proposition 2.4, one can find a sequence of CC sets associated with better CC policies such that at least one of these sets intersects $(\partial\Omega_{FR})^+$.

Let $I(\Omega)$ be the set whose elements are the corner points of Ω and $|I(\Omega)|$ its cardinality. We order such points increasingly with respect to their first coordinate, observing that for any two such successive corner points $(\alpha^{(i)}, \beta^{(i)})$ and $(\alpha^{(i+1)}, \beta^{(i+1)})$, the coordinate-convexity of Ω implies $\beta^{(i)} > \beta^{(i+1)}$. As $\Omega \cap (\partial\Omega_{FR})^+ = \emptyset$, Ω has at least two corner points, where the first one $(\alpha^{(1)} = 0, \beta^{(1)} > 0)$ is on the n_2-axis and the last one $(\alpha^{(|I(\Omega)|)} > 0, \beta^{(|I(\Omega)|)} = 0)$ is on the n_1-axis (see Figure 2.1(a)).

(a) If $|I(\Omega)| > 2$, then we apply Proposition 2.4 to the corner point $(\alpha^{(|I(\Omega)|)} > 0, \beta^{(|I(\Omega)|)} = 0)$, choosing $n = \alpha^{(|I(\Omega)|)} - \alpha^{(|I(\Omega)|-1)} - 1$, $p = \beta^{(|I(\Omega)|-1)} -$

$\beta^{(|I(\Omega)|)} - 1$, and m the largest nonnegative integer such that $(\alpha^{(|I(\Omega)|)} + m, \beta^{(|I(\Omega)|-1)} - \beta^{(|I(\Omega)|)} - 1) \in (\partial\Omega_{FR})^+$ (see Figure 2.1(b)). By Proposition 2.4, at least one of the inequalities $J(\Omega \setminus S^-) > J(\Omega)$ and $J(\Omega \cup S^+) > J(\Omega)$ holds. By construction, $(\Omega \cup S^+) \cap (\partial\Omega_{FR})^+ \neq \emptyset$, so if $J(\Omega \cup S^+) > J(\Omega)$ the statement is proved (see Figure 2.1(c)). Otherwise, $J(\Omega\setminus S^-) > J(\Omega)$. Note that $\Omega\setminus S^-$ does not intersect $(\partial\Omega_{FR})^+$ and has only $|I(\Omega)| - 1$ corner points (see Figure 2.1(d)), where the last one is $(\alpha^{(|I(\Omega)|-1)}, 0)$, where $\alpha^{(|I(\Omega)|-1)} > 0$. So, we can repeat the arguments used above starting from $\Omega \setminus S^-$ instead of Ω. After at most $|I(\Omega)| - 2$ applications of Proposition 2.4 we reach one of the following cases: either we find a CC policy better than the initial one and with associated CC set intersecting $(\partial\Omega_{FR})^+$, or we end up with the next case (b).

(b) If $|I(\Omega)| = 2$, then Ω is rectangular. Hence, the CAC system performs a decoupling between the two classes of users. Then Ω can be improved by extending one of its opposite sides until it meets $(\partial\Omega_{FR})^+$ (this corresponds to assigning more resources to one class of users, while maintaining the decoupling and not reducing the number of resources assigned to the other class). This can also be checked by noting that in this case $J(\Omega)$ has the expression (2.15) with $a = c = 0$ and the functions $x_k(\cdot, \cdot)$ in (2.27) are nondecreasing in each of their second arguments.

■

The proof may be obtained by using Proposition 2.5, instead of Proposition 2.4, or a combination of the two.

Given a generic region Ω, in the following we shall use the notations

$$l_1^{\Omega}(n_2) := \max\{j_1 \in \mathbb{N}_0 \text{ such that } (j_1, n_2) \in \Omega\}\,, \tag{2.18}$$

$$l_2^{\Omega}(n_1) := \max\{j_2 \in \mathbb{N}_0 \text{ such that } (n_1, j_2) \in \Omega\}\,. \tag{2.19}$$

The values $l_1^{\Omega}(n_2)$ and $l_2^{\Omega}(n_1)$ represent the maximum number of type-1/type-2 connections allowed in Ω when one has already n_2 type-2/n_1 type-1 connections, respectively. It follows from the definitions that the functions $l_k^{\Omega}(\cdot)$ are nonincreasing. Set $n_{1,\max}^{FR} := l_1^{\Omega_{FR}}(0)$, $n_{2,\max}^{FR} := l_2^{\Omega_{FR}}(0)$.

Proposition 2.8. *The following hold.*

(i) Let (α, β) be a type-2 corner point of Ω for which Proposition 2.4 cannot be applied. Then $l_2^{\Omega}(\alpha - 1) > l_2^{\Omega_{FR}}(\alpha)$.

(ii) Let (α, β) be a type-1 corner point of Ω for which Proposition 2.5 cannot be applied. Then $l_1^{\Omega}(\beta - 1) > l_1^{\Omega_{FR}}(\beta)$.

Proof of Proposition 2.4. We prove (i); for (ii), similar arguments can be used. The sets S^+ and S^- in Proposition 2.4 are rectangles with the same height p. The only value of p for which S^- is IR_{Ω} is $p = l_2^{\Omega}(\alpha - 1)$. The maximum possible value of p for which S^+ is IA_{Ω} is $p = l_2^{\Omega_{FR}}(\alpha)$. So, if $l_2^{\Omega}(\alpha - 1) > l_2^{\Omega_{FR}}(\alpha)$, then Proposition 2.4 cannot be applied. If, instead, $l_2^{\Omega}(\alpha - 1) \leq l_2^{\Omega_{FR}}(\alpha)$, then one can find sets S^- and S^+ that satisfy its assumptions (e.g., $p = l_2^{\Omega}(\alpha - 1)$, $n = m = 0$). ∎

Our next Proposition 2.9, which extends to nonlinearly-constrained feasibility regions the property stated in [34, Theorem 1] for the linear case, defines all the possible positions of the corner points of an optimal CC policy Ω^o. In particular, it states that the corner points of Ω^o can be located only among the vertices of a suitable grid (see Figure 2.3).

Proposition 2.9. *(i) If (α, β) is a type-2 corner point for Ω^o, then for some $j_2 = 1, \dots, n_{2,\max}^{FR}$ one has*

$$\alpha = l_1^{\Omega_{FR}}(j_2) + 1 \,. \tag{2.20}$$

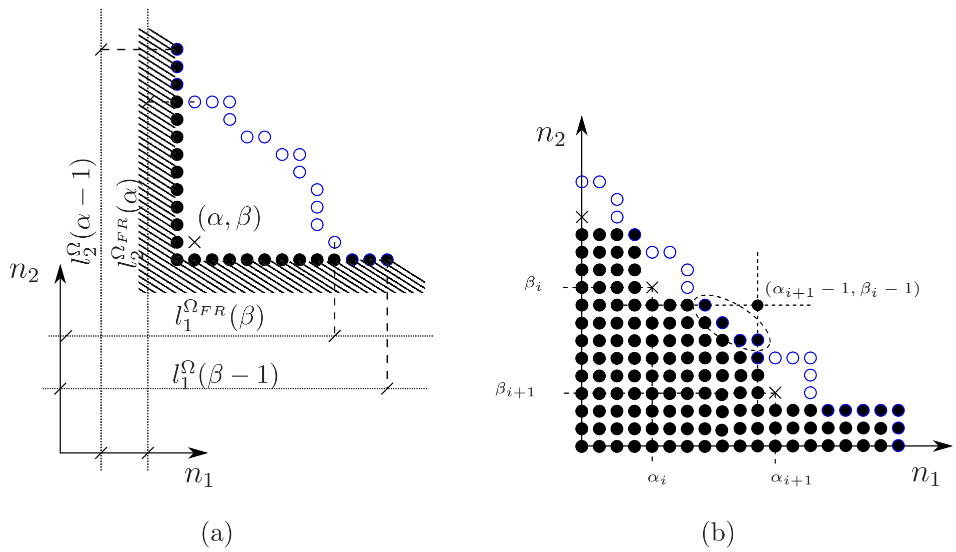


Figure 2.2: (a) An example of a corner point (α, β) for which neither Proposition 2.4 nor 2.5 can be applied. (b) An example of a CC policy Ω^o that cannot be improved by applying Propositions 2.4 and 2.5.

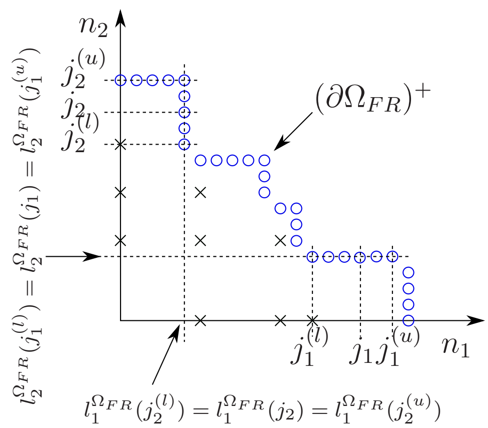


Figure 2.3: Potential locations of the corner points of an optimal CC policy Ω^o. According to Proposition 2.9, the corner points of Ω^o have to be searched among the points denoted by crosses.

(ii) If (α, β) is a type-1 corner point for Ω^o, then for some $j_1 = 1, \ldots, n_{1,\max}^{FR}$ one gets

$$\beta = l_2^{\Omega_{FR}}(j_1) + 1\,. \qquad (2.21)$$

The proof of Proposition 2.9 is obtained using the following technical lemma.

Lemma 2.10. From [34, Lemma 2] - *Let Ω^o be an optimal policy. Then: 1) if S is IA_{Ω^o}, then $J(S) \leq J(\Omega^o)$; 2) if S is IR_{Ω^o}, then $J(S) \geq J(\Omega^o)$.*

Proof of Proposition 2.9. We prove only (i), as (ii) can be obtained in the same way by exchanging the roles of the two classes of users. Suppose that (2.20) is violated for every $j = 1, \ldots, n_{2,\max}^{FR}$. Choosing $n = l_2^{\Omega^o}(\alpha-1)-\beta \geq 0$, $S^-(n) = \{(\alpha - 1, \beta + i) : i = 0, \ldots, n\} \subseteq \Omega^o$, and $S^+(n) = \{(\alpha, \beta + i) : i = 0, \ldots, n\} \subseteq \Omega_{FR} \setminus \Omega^o$ (see Figure 2.4), it follows that the sets $\Omega^o \setminus S^-(n)$ and $\Omega^o \cup S^+(n)$ are CC, so $S^-(n)$ is IR_{Ω^o} and $S^+(n)$ is IA_{Ω^o}. By formula (2.15), we get $J(S^-(n)) = r_1(\alpha-1)+r_2x_2(\beta, \beta+n) < r_1\alpha+r_2x_2(\beta, \beta+n) = J(S^+(n))$, but this contradicts the optimality condition stated in Lemma 2.10, so one concludes that there exists some $j_2 = 1, \ldots, n_{2,\max}^{FR}$ such that (2.20) holds. ■

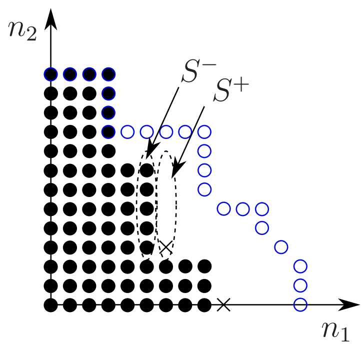


Figure 2.4: An example of a CC set Ω having a type-2 corner point (α, β) for which $\alpha \neq l_1^{\Omega_{FR}}(j_2) + 1$ for every $j_2 = 1, \ldots, n_{2,\max}^{FR}$.

Let $\mathcal{S}$ be the set of all CC policies whose corner points are among the vertices of a grid G with vertical lines of equations $\alpha = 0$ and $\alpha = l_1^{\Omega_{FR}}(j_2)+1$ ($j_2 = 1, \ldots, n_{2,\max}^{FR}$) and horizontal lines of equations $\beta = 0$ and $\beta = l_2^{\Omega_{FR}}(j_1)+1$ ($j_1 = 1, \ldots, n_{1,\max}^{FR}$). According to Proposition 2.9, any optimal CC policy Ω^o belongs to $\mathcal{S}$.

The next Proposition 2.11, provides a characterization of the upper boundary $(\partial\Omega^o)^+$ of Ω^o and its intersection with $(\partial\Omega_{FR})^+$, when Ω^o has at least two corner points. Theorem 2.11 implies that between any two successive corner points the intersection between $(\partial\Omega^o)^+$ and $(\partial\Omega_{FR})^+$ is nonempty (see the dotted ellipse in Figure 2.2(b)). Up to our knowledge, no such result was previously available in the literature, even for linearly-constrained feasibility regions.

Proposition 2.11. *Let (α_i, β_i) and $(\alpha_{i+1}, \beta_{i+1})$ be two consecutive corner points of Ω^o. Then the intersection between the vertical line $n_1 = \alpha_{i+1} - 1$ and the horizontal line $n_2 = \beta_i - 1$ either lies on $(\partial\Omega_{FR})^+$ or is outside Ω_{FR}.*

Proof of Proposition 2.11. The claim is equivalent to the pair of inequalities

$$l_1^{\Omega_{FR}}(\beta_i - 1) \leq \alpha_{i+1} - 1\,, \tag{2.22}$$

$$l_2^{\Omega_{FR}}(\alpha_{i+1} - 1) \leq \beta_i - 1\,. \tag{2.23}$$

Let us prove (2.22). By the definition of $l_2^{\Omega^o}(\alpha_i)$, the monotonicity of $l_2^{\Omega^o}(\cdot)$, and Proposition 2.8 (i), we get

$$\beta_i - 1 = l_2^{\Omega^o}(\alpha_i) \geq l_2^{\Omega^o}(\alpha_{i+1} - 1) > l_2^{\Omega_{FR}}(\alpha_{i+1})\,. \tag{2.24}$$

Now, suppose that the inequality $l_1^{\Omega_{FR}}(\beta_i - 1) > \alpha_{i+1} - 1$, opposite to (2.22), holds. Let us show that this leads to a contradiction. As α_{i+1} is an integer, one has

$$l_1^{\Omega_{FR}}(\beta_i - 1) > \alpha_{i+1} - 1 \Leftrightarrow l_1^{\Omega_{FR}}(\beta_i - 1) \geq \alpha_{i+1}\,.$$

This, combined with the property $l_2^{\Omega_{FR}}(l_1^{\Omega_{FR}}(\beta_i - 1)) \geq \beta_i - 1$ (which is a consequence of (2.19) and (2.18)) and the monotonicity of $l_2^{\Omega_{FR}}(\cdot)$, implies $l_2^{\Omega_{FR}}(\alpha_{i+1}) \geq \beta_i - 1$, but this contradicts (2.24). So, (2.22) must hold. The proof of (2.23) is similar. ∎

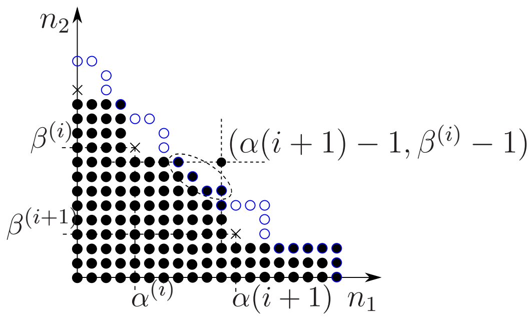


Figure 2.5: Characterization of an optimal CC policy with at least 2 corner points, according to Proposition 2.11.

Proposition 2.7 can be obtained as a consequence of Proposition 2.11 by observing that a CC policy that has no corner points coincides with the complete sharing policy and a CC policy that has only one corner point can have only one of the three shapes shown in Figure 2.6 for which there is a nonempty intersection with $(\partial\Omega_{FR})^+$.

According to the next Proposition 2.12, in order to identify a CC policy and, in particular, to find an optimal one, it is sufficient to search within the set of corner points. More specifically, the proposition states that one can construct a CC policy starting merely from the knowledge of the positions of its corner points.

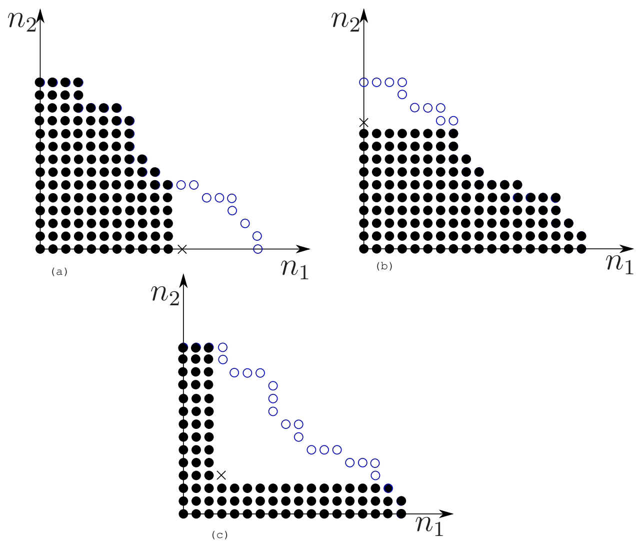


Figure 2.6: The three possible shapes of a CC set with a unique corner point.

Proposition 2.12. *Let $I(\Omega)$ denote the set of corner points $\{(\alpha^{(i)}, \beta^{(i)})\}$ of a CC policy $\Omega \subseteq \Omega_{FR}$ and $C_i^- := \{\mathbf{n} \in \Omega_{FR} : n_1 \geq \alpha^{(i)} \text{ and } n_2 \geq \beta^{(i)}\}$. Then $\Omega = (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-)$.*

Proof of Proposition 2.12. We build Ω starting from Ω_{FR} and removing subregions from Ω_{FR}, each associated with one of the $|I(\Omega)|$ corner points of Ω. To simplify the notations, in the proof we assume $I(\Omega) \neq \emptyset$. The case $I(\Omega) = \emptyset$ can be dealt with in a similar way.

Let $(\alpha^{(i)}, \beta^{(i)})$ be one of these corner points. Consider a point $(\hat{n}_1, \hat{n}_2) \in C_i^- := \{\mathbf{n} \in \Omega_{FR} : n_1 \geq \alpha^{(i)} \text{ and } n_2 \geq \beta^{(i)}\}$ and suppose that $(\hat{n}_1, \hat{n}_2) \in \Omega$. Then, the coordinate-convexity of Ω implies $(\hat{n}_1, \beta^{(i)}) \in \Omega$ and then $(\alpha^{(i)}, \beta^{(i)}) \in \Omega$, which is a contradiction. So, each corner point $(\alpha^{(i)}, \beta^{(i)})$ excludes from Ω_{FR} the set C_i^-. Then

$$\Omega \subseteq (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \,. \tag{2.25}$$

Now, we show that also $\Omega = (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-)$ holds, so Ω is completely determined by the knowledge of the locations of all its corner points. This can be proved by showing that if $(\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \setminus \Omega \neq \emptyset$, then Ω would have at least $|I(\Omega)|+1$ corner points, which is a contradiction. We detail this part of the proof considering what happens inside the strip $Q_i := \{(n_1, n_2) \in \mathbb{N}_0^2 : \alpha^{(i)} \leq n_1 < \alpha^{(i+1)}\}$ between two consecutive corner points $(\alpha^{(i)}, \beta^{(i)})$ and $(\alpha^{(i+1)}, \beta^{(i+1)})$ (suppose for simplicity of notation that all their coordinates are positive)[1]. First of all, note that $(\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i = (\Omega_{FR} \setminus C_i^-) \cap Q_i$. Then, by the definition of a corner point and the coordinate-convexity of Ω one has $V_i := \{(\alpha^{(i)}, k) : k = 0, \ldots, \beta^{(i)} - 1\} \subseteq \Omega$ and $H_i := \{(\alpha^{(i)} + h, j) : h = 0, \ldots, \alpha^{(i+1)} - \alpha^{(i)} - 1, j = 0, \ldots, \beta^{(i+1)}\} \subseteq \Omega$.

Let $E_i := ((\Omega_{FR} \setminus C_i^-) \cap Q_i) \setminus (V_i \cup H_i)$, $F_i := E_i \setminus \Omega$, and suppose that $F_i \neq \emptyset$. Let (p_1, p_2) be one of the points of F_i with minimal first coordinate, then by the coordinate-convexity of Ω one has $\{(p_1 - 1, l) : l = 0, \ldots, p_2\} \subseteq \Omega$. Then, let $(p_1, \hat{p}_2)$ be the only point of F_i with first coordinate p_1 and minimal second coordinate. One can check by the definition that $(p_1, \hat{p}_2)$ is a corner point, but this is a contradiction, since the only corner point in the strip Q_i is $(\alpha^{(i)}, \beta^{(i)})$ by construction. Then the set F_i must be empty and, therefore, $E_i \subseteq \Omega$.

Summing up, $(\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i \subseteq \Omega \cap Q_i$. This, combined with $\Omega \cap Q_i \subseteq (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i$ (obtained from (2.25)) proves that $\Omega \cap Q_i = (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i$. Similarly, for the sets Q_0 and $Q_{|I(\Omega)|}$ defined in footnote 1, one has $\Omega \cap Q_0 = (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_0$, and $\Omega \cap Q_{|I(\Omega)|+1} = (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_{|I(\Omega)|+1}$. Hence, $\Omega = (\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-)$.

■

[1] Similarly, if $\alpha^{(1)} > 0$, then one should also consider the first strip $Q_0 := \{(n_1, n_2) \in \mathbb{N}_0^2 : 0 \leq n_1 < \alpha^{(1)}\}$, and if $\beta^{(|I(\Omega)|)} > 0$, then one should take into account also the last strip $Q_{|I(\Omega)|+1} := \{(n_1, n_2) \in \mathbb{N}_0^2 : n_1 \geq \alpha^{(|I(\Omega)|)}\}$.

Proposition 2.12 shows that a CC policy Ω can be always obtained by removing particular regions C_i from the feasibility region Ω_{FR}. C_i regions are built by using the corner points of Ω.

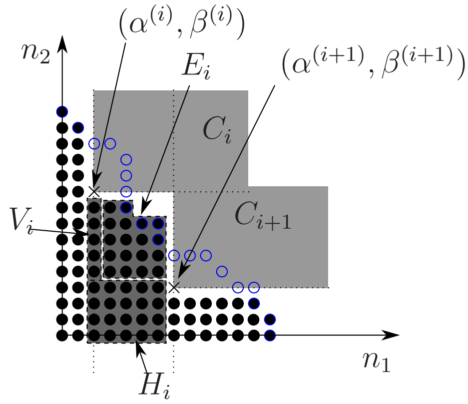


Figure 2.7: Sets V_i, H_i and E_i considered in the proof of Proposition 2.12.

2.1.3 Initialization of Algorithm 1

The structural properties of the optimal CC policies can be exploited sometimes also for the initialization of Algorithm 1. For instance, the approach described in the previous paragraph to find the optimal CC policies may be useful also for a feasibility region characterized by a more complex upper boundary $(\partial\Omega_{FR})^+$ (i.e., when n_{rect} is large). Indeed, in such a case one may replace the feasibility region Ω_{FR} by another one $\Omega'_{FR} \subset \Omega_{FR}$, characterized by a much smaller value of n_{rect} (say n'_{rect}). Then, one may find an optimal CC policy Ω'^o for the CAC problem with feasibility region Ω'_{FR} (this would be a much easier task than for the original problem, since $n'_{\text{rect}} \ll n_{\text{rect}}$). Then, since $\Omega'^o \subseteq \Omega'_{FR} \subset \Omega_{FR}$, Ω'^o may be chosen as the initial policy Ω_1 for Algorithm 1 for the problem with the original feasibility region Ω_{FR}. Another reasonable choice for the initial policy may be $\Omega_1 := \Omega'_{FR,L}$, where $\Omega'_{FR,L}$

is a feasibility region obtained by linearizing the nonlinear upper boundary $(\partial\Omega_{FR})^+$ of the original feasibility region Ω_{FR} (thus reducing the generalized stochastic knapsack to its linearized version, studied in [34], and taking the so-called complete sharing policy for such problem), or $\Omega_1 := \Omega_L'^o$, where $\Omega_L'^o$ is the optimal policy for the problem with feasibility region $\Omega'_{FR,L}$, which in certain cases can be found by applying the structural results for linearly-constrained feasibility regions obtained in [34].

2.1.4 Cardinality of the set of candidate optimal CC policies

The results described in Section 2.1.2 can be applied to narrow the search for the optimal CC policies to the ones that satisfy the necessary conditions stated in Propositions 2.11, 2.7, and 2.9. For instance, for the feasibility region depicted in Figure 2.8, Table 2.1 shows how the number of candidate optimal CC policies decreases as each of the previous necessary optimality conditions is added (the remaining of this section describes how the numbers in Table 2.1 were obtained).

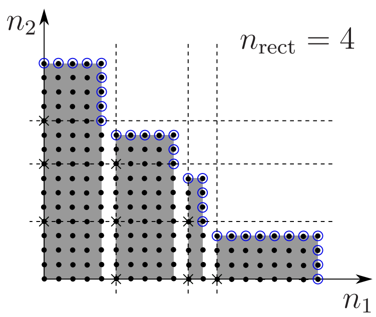


Figure 2.8: Decomposition of the feasibility region into discrete disjoint rectangles. The crosses represent the potential locations of the corner points of an optimal CC policy, according to Proposition 2.9.

Number of CC policies	
All policies	> 352715
Policies satisfying Proposition 2.9	42
Policies satisfying Propositions 2.7 and 2.9	28
Policies satisfying Propositions 2.11, 2.7, and 2.9	15

Table 2.1: Number of CC policies satisfying the constraints coming from the necessary optimality conditions stated in Propositions 2.11, 2.7, and 2.9.

This approach allows one to find the optimal CC policies with a low computational burden when the grid satisiying conditions (2.20) and (2.21) has a few number of vertices. As one can observe from Figure 2.8, this number depends only on the quantity n_{rect} (e.g., by the next Proposition 2.13 one obtains that this number is 42 for $n_{\mathrm{rect}} = 4$).

Note that the number of candidate optimal CC policies is a function of n_{rect} only and that in general is far less than $2^{|G|}$, where $|G|$ is the cardinality of the grid G. Indeed, if a candidate optimal CC policy has a corner point that is a vertex of G, then some other vertices of G are automatically excluded to be corner points for that policy. More precisely, for two successive corner points (α_i, β_i) and $(\alpha_{i+1}, \beta_{i+1})$ with $\alpha_i < \alpha_{i+1}$, the coordinate-convexity of the policy imposes the constraint

$$\beta_i > \beta_{i+1} \,. \tag{2.26}$$

In particular, this implies that any CC policy has at most

$$n_{\mathrm{rect}} \le \min\{n_{1,\max}^{FR}, n_{2,\max}^{FR}\}$$

corner points. It also implies that, if a CC policy has a corner point (α, β), then it cannot have other corner points inside the rectangle of vertices $(0,0), (\alpha,0), (\alpha,\beta), (0,\beta)$.

Note also that if the potential corner points are chosen such the previous constraints (2.26) are satisfied, then the set defined as $(\Omega_{FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-)$ is CC by construction (this can be checked, e.g., by applying Lemma 2.20).

The following proposition provides a way to compute the cardinality $|\mathcal{S}|$ of the set $\mathcal{S}$. In order to write its dependence on n_{rect}, we also denote such cardinality by $|\mathcal{S}(n_{\text{rect}})|$.

Proposition 2.13. *For $n_{\text{rect}} \geq 1$, one has $|\mathcal{S}(n_{\text{rect}})| = \frac{1}{n_{\text{rect}}-1}\binom{2(n_{\text{rect}}-1)}{n_{\text{rect}}-1}$, or equivalently, $|\mathcal{S}(n_{\text{rect}})| = C_{n_{\text{rect}}-1}$, where, for n a nonnegative integer, C_n is the n-th Catalan number, defined as $C_n = \frac{1}{n}\binom{2n}{n}$.*

Proof of Proposition 2.13. We observe from Figure 2.8 that the number of CC policies with corner points on the grid G is equal to the number of CC subsets of the auxiliary feasibility region $\hat{\Omega}_{FR}$ shown in Figure 2.9. Indeed, the form of the grid G depends only on n_{rect} and not on the actual coordinates of its vertices (which instead do depend on the feasibility region). Now, it is easily seen from Figure 2.10 that, for $n = n_{\text{rect}} - 1$, the number of CC subsets of the auxiliary feasibility region $\hat{\Omega}_{FR}$ is equal to the number of different monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the NW-SE diagonal (we recall that a monotonic path is one which starts in the upper left corner, finishes in the lower right corner, and consists entirely of edges pointing rightwards or downwards). It is well-known (see, e.g., [49]) that such a number of monotonic paths is the n-th Catalan number C_n. So, one obtains $|\mathcal{S}(n_{\text{rect}})| = C_{n_{\text{rect}}-1}$.

■

As an example, by Proposition 2.13 one has $|\mathcal{S}(n_{\text{rect}})| = 1, 5, 14, 42$ for $n_{\text{rect}} = 1, 2, 3, 4$, respectively.

The number of candidate optimal CC policies can be further decreased by imposing their compatibility with the structural properties of the optimal

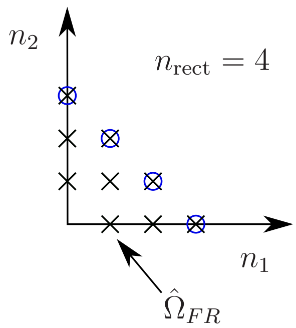


Figure 2.9: A feasibility region with the same grid G as the feasibility region shown in Figure 2.8.

CC policies described in Subsections 2.1.2 and 2.1.5. In particular, additional constraints on the locations of the corner points of an optimal policy follow from the general structural properties stated in Propositions 2.7 and 2.11.

For instance, as a consequence of Proposition 2.13, we obtain the following result.

Proposition 2.14. *For $n_{\mathrm{rect}} \geq 2$, the number of CC policies with corner points on the grid G (i.e., satisfying Proposition 2.9) and satisfying Proposition 2.7 is equal to*

$$|\mathcal{S}(n_{\mathrm{rect}})| - |\mathcal{S}(n_{\mathrm{rect}} - 1)| = C_{n_{\mathrm{rect}}-1} - C_{n_{\mathrm{rect}}-2} \,.$$

Proof of Proposition 2.14. Proceeding likewise in the proof of Proposition 2.13, one can assume without loss of generality that the feasibility region is the one $\hat{\Omega}_{FR}$ shown in Figure 2.9. Now, for such feasibility region, the number of CC policies with corner points on the grid G and intersecting its upper boundary is obtained by subtracting from the total number of policies $C_{n_{\mathrm{rect}}-1}$ determined by Proposition 2.13, the number of CC policies with corner points on the subgrid G' shown in Figure 2.11. Since G' has the same form as G

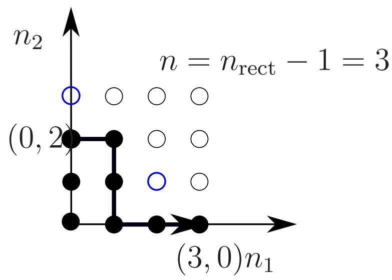


Figure 2.10: A monotonic path from the point $(0, 2)$ to the one $(3, 0)$. It is associated with the CC subset $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(3,0)\}$ of the feasibility region shown in Figure 2.9.

(with n_{rect} replaced by $n_{\text{rect}} - 1$), such a number can be still computed by an application of Proposition 2.13, so it is equal to $C_{n_{\text{rect}}-2}$.

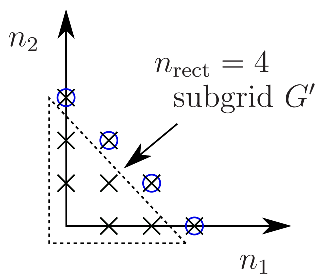


Figure 2.11: The subgrid G' for the feasibility region shown in Figure 2.9.

■

Similarly, the following result shows the number of all CC policies with corner points on the grid G and satisfying Propositions 2.11 and 2.7.

Proposition 2.15. *For $n_{\text{rect}} \geq 2$, the number of CC policies with corner points on the grid G (i.e., satisfying Proposition 2.9) and satisfying Propositions 2.11 and 2.7 is equal to $2^{n_{\text{rect}}} - 1$.*

Proof of Proposition 2.15. Proceeding likewise in the proofs of Propositions 2.13 and 2.14, we refer to the feasibility region $\hat{\Omega}_{FR}$ shown in Figure 2.9. Each CC policy with its corner points on the grid G and satisfying Propositions 2.11 and 2.7 can be represented by a monotonic path starting from the point $(-1, n_{\text{rect}})$ and ending in the point $(n_{\text{rect}}, -1)$ (see Figure 2.12). Such a monotonic path is characterized by the fact that it starts downwards and each time a number of steps downwards is made, the same number of steps rightwards is made as the path changes direction. It can be easily seen that this choice of the monotonic path implies that both requirements coming from Propositions 2.11 and 2.7 are satisfied. Moreover, the particular form of such a monotonic path implies that the path can be represented by its ordered sequence of steps downwards alone, e.g., $(2, 1, 2)$ in Figure 2.12. In general, each such monotonic path can be represented by a sequence $(l_1, \ldots, l_h)$, where $l_1, \ldots, l_h$ are $h \le n_{\text{rect}} + 1$ positive integers, and $\sum_{i=1}^{h} l_i = n_{\text{rect}} + 1$ (the only case that does not correspond to a CC policy satisfying Proposition 2.7 is the sequence of all 1's). We recall that, for each positive integer n, the number $P(n)$ of distinct ordered sequences $(l_1, \ldots, l_h)$ with $h \le n$ such that $\sum_{i=1}^{h} l_i = n$ is equal to 2^{n-1}. Indeed, a dynamic programming argument shows that, setting $P(0) = 1$, one has $P(n) = \sum_{k=1}^{n} P(n-k)$, from which one concludes by induction that $P(n) = 2^n - 1$. Concluding, the number of CC policies with corner points on the grid G and satisfying Propositions 2.11 and 2.7 is equal to $2^{n_{\text{rect}}} - 1$.

■

It is interesting to compare the expression of $|\mathcal{S}|$ given in Proposition 2.13 with the cardinality of the set $\mathcal{S}_C$ of all CC subsets of Ω_{FR} (i.e., of all CC policies), without imposing the constraint that their corner points are among the vertices of the grid G. Let us start by considering the case of a

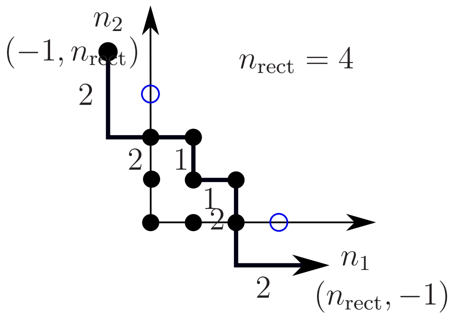


Figure 2.12: The monotonic path $(2, 2, 1)$ associated with the CC subset $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1)\}$ of the feasibility region shown in Figure 2.9.

particularly simple feasibility region.

Proposition 2.16. *Let Ω_{FR} be rectangular. Then*

$$|\mathcal{S}_C| = \frac{(n_{1,\max}^{FR} + n_{2,\max}^{FR} + 2)!}{(n_{1,\max}^{FR} + 1)!(n_{2,\max}^{FR} + 1)!} - 1\,.$$

Proof of Proposition 2.16. For Ω_{FR} a rectangular feasibility region, each CC set $\Omega \subseteq \Omega_{FR}$ can be represented by a monotonic path starting from the point $(-1, l_2^{\Omega}(0))$ and ending in the point $(l_1^{\Omega}(0), -1)$ (see Figure 2.13 for an example). Such a monotonic path has a total number of $l_2^{\Omega}(0) + 1$ steps downwards at locations taken from the set $\{0, \dots, n_{1,\max}^{FR}\}$, where the same location can appear more than once (for instance, the monotonic path represented in Figure 2.13 has 1 step downwards for $n_1 = 1$, 2 steps for $n_2 = 2$, and 1 step for $n_1 = 4$). So, each CC set $\Omega \subseteq \Omega_{FR}$ with a given $l_2^{\Omega}(0)$ can be associated in a one-to-one way with a combination with repetition of k elements from the set $\{0, \dots, n_{1,\max}^{FR}\}$ of cardinality $n = n_{1,\max}^{FR} + 1$, where $k = l_2^{\Omega}(0) + 1 \in \{1, \dots, n_{2,\max}^{FR} + 1\}$. The number of such different combinations is denoted by $\left\langle {n \atop k} \right\rangle = \binom{n+k-1}{k}$ and is equal to $\frac{(n+k-1)!}{k!(n-1)!}$ [50, p. 16]. So, $|\mathcal{S}_C| = \sum_{k=1}^{n_{2,\max}^{FR}+1} \left\langle {n_{1,\max}^{FR}+1 \atop k} \right\rangle$. Then one exploits the relation $\sum_{k=1}^{h} \left\langle {n \atop k} \right\rangle = \left\langle {n+1 \atop h} \right\rangle - 1$

from [50, p. 16], and obtains $|\mathcal{S}_C| = \left\langle {n_{1,\max}^{FR}+2 \atop n_{2,\max}^{FR}+1} \right\rangle = \frac{(n_{1,\max}^{FR}+n_{2,\max}^{FR}+2)!}{(n_{1,\max}^{FR}+1)!(n_{2,\max}^{FR}+1)!} - 1$.

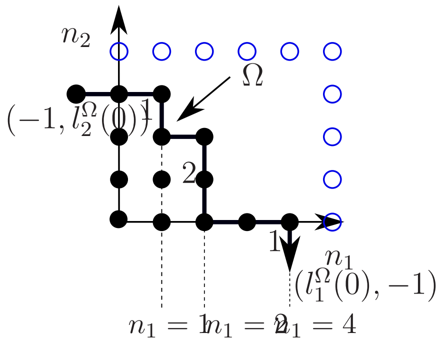


Figure 2.13: The monotonic path associated with the CC subset $\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(3,0),(4,0)\}$ of the rectangular feasibility region $\{0,1,\ldots,5\} \times \{0,1,\ldots,4\}$.

■

It follows from their definitions that one has always $|\mathcal{S}_C| \geq |\mathcal{S}|$. This is the way we estimated the number of all CC subsets of Ω_{FR} in Table 2.1. For the case of a rectangular feasibility region, $|\mathcal{S}_C|$ can be very large, whereas $|\mathcal{S}| = P_1(0,0) = 1$ holds always. Similarly, in the more general case of a feasibility region with $\min\{n_{1,\max}^{FR}, n_{2,\max}^{FR}\} \gg n_{\text{rect}}$, Propositions 2.13 and 2.16 imply $|\mathcal{S}_C| \gg |\mathcal{S}|$. For instance, the lower bound 352715 on the number of all CC policies in the example in Table 2.1 was obtained by applying Proposition 2.16 to the rectangular subregion $\{0,1,\ldots,9\} \times \{0,1,\ldots,10\}$ of its feasibility region.

2.1.5 Structural properties dependent on the revenue ratio

Let us now consider structural properties of the optimal CC policies obtained for suitable values of the *revenue ratio* $R := r_2/r_1$.

Three definitions are necessary:

Definition 2.17. [From [34]] - *We define*

$$x_k(a,b) := \frac{\sum_{j=a}^{b} j q_k(j)}{\sum_{j=a}^{b} q_k(j)}\,, \tag{2.27}$$

with $q_k(j)$ defined in (2.4). $x_k(a,b)$ represents the expected value of the random variable $X_k(a,b)$ corresponding to the equilibrium state of a birth-death process with birth rates $\bar{\lambda}_k(j)$ and $j \geq 0$ given by

$$\bar{\lambda}_k(j) = \begin{cases} \lambda_k(j), & j < b \\ 0, & j = b \end{cases}$$

and death rates $\bar{\mu}_k(j)$ and $j \geq 1$ given by

$$\bar{\mu}_k(j) = \begin{cases} j\mu_k(j), & j > a \\ 0, & j = a. \end{cases}$$

Definition 2.18. *Let B_1 and B_2 be, respectively, the maximum width and the maximum height of a step in the upper boundary of the feasibility region $(\partial\Omega_{FR})^+$. Formally,*

$$\begin{aligned} B_1 &:= \max\{j_1^{(u)} - j_1^{(l)} + 1 : j_1 = 0, \dots, n_{1,\max}^{FR}\} \leq n_{1,\max}^{FR} + 1\,, \\ B_2 &:= \max\{j_2^{(u)} - j_2^{(l)} + 1 : j_2 = 0, \dots, n_{2,\max}^{FR}\} \leq n_{2,\max}^{FR} + 1\,. \end{aligned} \tag{2.28}$$

Definition 2.19. *Let $k = 1, 2$. A CC policy Ω is threshold type-k if and only if for some $t_k = 0, \dots, n_{k,\max}$ we get*

$$\Omega = \{(n_1, n_2) \in \Omega_{FR} : n_k \leq t_k\}\,.$$

Lemma 2.20. *The following two statements are equivalent:*

(i) Ω is CC;

(ii) for $k = 1, 2$ and $t_k = 0, \ldots, n_{k,\max}$, consider the (one-dimensional) intersection I_{t_k} between Ω and the line of equation $n_k = t_k$. Denote by $\bar{k}$ the other index (i.e., if $i = k$, $\bar{k} = 2$ and if $k = 2$, $\bar{k} = 1$, vice-versa). Only one of the three following cases can happen (Figure 2.14):

(a) $I_{t_k} = \emptyset$;

(b) $I_{t_k} = \{0, \ldots, l_{\bar{k}}^{\Omega_{FR}}(t_k)\}$;

(c) there exists an integer $n_{\bar{k}}(t_k) \in [0, l_{\bar{k}}^{\Omega_{FR}}(t_k))$ *such that* $I_{t_k} = \{0, \ldots, n_{\bar{k}}(t_k)\}$.

Proof of Lemma 2.20. ($i \Rightarrow ii$) Let us consider the case $k = 1$ (the proof for $k = 2$ is similar). If the set I_{t_1} is nonempty, then let (t_1, p_2) be such that $p_2 \in I_{t_1}$. An upper bound on the maximum possible value of p_2 is obviously $l_2(t_1)$, otherwise (t_1, p_2) would be outside Ω_{FR}, and since $\Omega \subseteq \Omega_{FR}$, it would be outside Ω, too. By the coordinate-convexity of Ω, if $p_2 > 0$ then $(t_1, p_2 - 1) \in \Omega$, so $p_2 - 1 \in I_{t_1}$. Then, by backward induction on p_1, only the three cases described in statement ii are possible.

($ii \Rightarrow i$) Let $\mathbf{n} \in \Omega$ be such that at least one of its coordinates is greater than 0, suppose this is n_1. Then $n_1 \in I_{t_2}$ with $t_2 = n_2$. So one of cases (iib) or (iic) shows up, then $n_1 - 1 \in I_{t_2}$, or equivalently $(n_1 - 1, n_2) \in \Omega$, so Ω is CC. ■

Proposition 2.21. *For a nonlinearly constrained feasibility region Ω_{FR}, a CC policy is a type-k threshold if and only if it has no type-k corner points ($k = 1, 2$).*

Proof of Proposition 2.21. Let us consider the case $k = 1$ (the proof for $k = 2$ is similar).

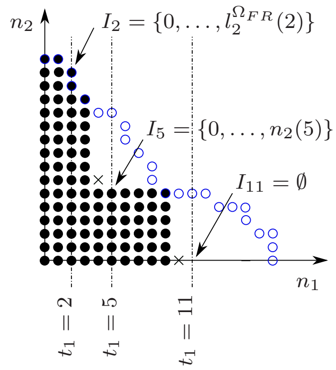


Figure 2.14: An example of the three cases described in Lemma 2.20.

- (*Only if*) Let Ω be a threshold type-1 policy. Then

$$(p_1, p_2) \in \Omega \Rightarrow (p_1, i) \in \Omega, \forall\, i \in \{0, \dots, l_2^{\Omega}(p_1)\}\,. \tag{2.29}$$

  Now, let $(\alpha, \beta) \in (\Omega_{FR} \setminus \Omega)$ be a candidate type-1 corner point. Then $(\alpha, \beta - 1) \in \Omega$ by the definition of a type-1 corner point. However, it follows from (2.29) that $(\alpha, \beta) \in \Omega$, a contradiction. So there do not exist type-1 corner points.

- (*If*) For $t_1 = 0, \dots, n_{1,\max}^{FR}$, consider the (one-dimensional) intersection I_{t_1} between Ω and the vertical line of equation $n_1 = t_1$. Due to the coordinate-convexity of Ω, by Lemma 2.20 only one of the three following cases can happen:

  (a) $I_{t_1} = \emptyset$;

  (b) $I_{t_1} = \{0, \dots, l_2^{\Omega_{FR}}(t_1)\}$;

  (c) $I_{t_1} = \{0, \dots, l_2^{\Omega}(t_1)\}$, with $l_2^{\Omega}(t_1) < l_2^{\Omega_{FR}}(t_1)$.

  It is easy to see by the coordinate-convexity of Ω that if only cases (a) and (b) can show up, then Ω is a threshold type-1 policy. Let us show that, if Ω has no type-1 corner points, then case (c) cannot happen (so Ω is a threshold type-1 policy because of the statement above).

Suppose now that case (c) happens, so that there exists $t_1^* \in [0, \ldots, n_{1,\max}^{FR}]$ and an integer $l_2^{\Omega}(t_1^*) \in [0, l_2^{\Omega_{FR}}(t_1^*))$ such that $I_{t_1^*} = \{0, \ldots, l_2^{\Omega}(t_1^*)\}$. Then the tuple $(t_1^*, l_2^{\Omega}(t_1^*) + 1) \in \Omega_{FR}$, but $(t_1^*, l_2^{\Omega}(t_1^*) + 1) \notin \Omega$, and this is not a type-1 corner point by assumption. Suppose $t_1^* > 0$, then it follows that it must be $(t_1^* - 1, l_2^{\Omega}(t_1^*) + 1) \notin \Omega$ (otherwise $(t_1^*, l_2^{\Omega}(t_1^*) + 1)$ would be a type-1 corner point), whereas $(t_1^* - 1, l_2^{\Omega}(t_1^*)) \in \Omega$ because $(t_1^*, l_2^{\Omega}(t_1^*)) \in \Omega$ and Ω is CC (Figure 2.15). Then by Lemma 2.20 $I_{t_1^*-1}$ is still of the form $I_{t_1^*-1} = \{0, \ldots, l_2^{\Omega}(t_1^* - 1)\}$, with $l_2^{\Omega}(t_1^* - 1) = l_2^{\Omega}(t_1^*) < l_2^{\Omega_{FR}}(t_1^* - 1)$ (note that $l_2^{\Omega_{FR}}(t_1^* - 1) \geq l_2^{\Omega_{FR}}(t_1^*)$ since the function $l_2^{\Omega_{FR}}(\cdot)$ is nonincreasing by definition). So, by backward induction on t_1^*, we arrive at $t_1^* = 0$. However, if $t_1^* = 0$, then it is easy to check that the tuple $(0, l_2^{\Omega}(0) + 1)$ satisfies the definition of a type-1 corner point, a contradiction. So we conclude that case (c) cannot happen, and Ω is a threshold type-1 policy.

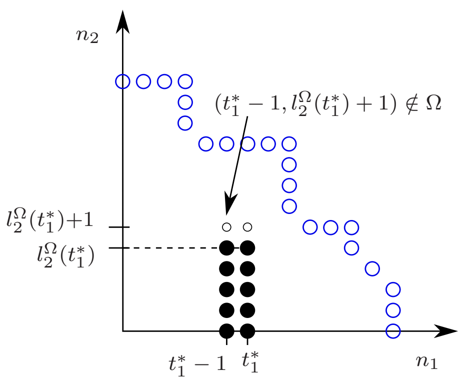


Figure 2.15: A description of a step in the proof of Proposition 2.21.

■

The following theorem states that under suitable conditions one has threshold-type optimal policies. The main idea of the proof consists in deriving conditions that a point of the grid G has to satisfy in order to be a corner point for Ω^o (see Lemma 2.25). When such conditions are not satisfied for a suitable subset of the grid, then Ω^o is of threshold type (see Lemma 2.26). The result is an extension of [34, Theorem 1] to feasibility regions with a nonlinear upper boundary. As we said, an extension of [34, Theorem 1] for a nonlinearly-constrained feasibility region less general than ours and under a different assumption on the holding-time distribution of the calls was reported in [45, Section 4].

Theorem 2.22. *Let $\lambda_k(\cdot)$ be nonincreasing for $k = 1, 2$ and $R := r_2/r_1$.*
(i) If $R > x_1(0, B_1)$, then Ω^o is threshold type-1, and the threshold is equal to some $l_1^{\Omega_{FR}}(j_2)$ for some $j_2 = 0, \dots, n^{FR}_{2,\max}$.

(ii) If $\frac{1}{R} > x_2(0, B_2)$, then Ω^o is threshold type-2, and the threshold is equal to some $l_2^{\Omega_{FR}}(j_1)$ for some $j_1 = 0, \dots, n^{FR}_{1,\max}$.

(iii) If $x_1(0, B_1) < R < \frac{1}{x_2(0,B_2)}$, then $\Omega^o = \Omega_{FR}$.

The proof of Theorem 2.22 is obtained combining the following technical lemmas.

Lemma 2.23. [From [34, Lemma 1]] - *Let $K = 2$. A policy is double-threshold if and only if it has no corner points. A policy is a type-k threshold if and only if it has no type-k corner points ($k = 1, 2$).*

Lemma 2.24. [From [34, Lemma 3]] - *For any nonnegative integers a, b, c, d, e, f with $b \geq a, d \geq c, f \geq e, b + d \geq f, a + c \geq e$, we have $x_k(a, b) + x_k(c, d) \geq x_k(e, f)$; $k = 1, \dots, K$.*

Lemma 2.25 is our extension of [34, Lemma 4] to general nonlinearly-constrained feasibility regions. With respect to [34], due to the different shape of the feasibility region, in general, it is not true that $j \neq h$ implies $l_1^{\Omega_{FR}}(j) \neq l_1^{\Omega_{FR}}(h)$. As shown in Figure 2.3, for every $j_2 \in \{0, \ldots, n_{2,\max}^{FR}\}$ there exist a minimum index $j_2^{(l)} \leq j_2$ and a maximum index $j_2^{(u)} \geq j_2$ such that $l_1^{\Omega_{FR}}(\cdot)$ is constant on the set $\{j_2^{(l)}, \ldots, j_2^{(u)}\} \subseteq \{0, \ldots, n_{2,\max}^{FR}\}$. Similarly, for every $j_1 \in \{0, \ldots, n_{1,\max}^{FR}\}$ there exist a minimum index $j_1^{(l)} \leq j_1$ and a maximum index $j_1^{(u)} \geq j_1$ such that $l_2^{\Omega_{FR}}(\cdot)$ is constant on the set $\{j_1^{(l)}, \ldots, j_1^{(u)}\} \subseteq \{0, \ldots, n_{1,\max}^{FR}\}$.

Lemma 2.25. *(i) If (α, β) is a type-2 corner point for Ω^o and $\lambda_2(\cdot)$ is nonincreasing, then for some $j_2 = 1, \ldots, n_{2,\max}^{FR}$ (2.20) holds together with*

$$\begin{aligned} Rx_2(0, B_2) \geq\ & x_1(l_1^{\Omega_{FR}}(j_2^{(l)}) + 1, l_1^{\Omega_{FR}}(j_2^{(l)} - 1)) \\ & -x_1(l_1^{\Omega_{FR}}(j_2^{(u)} + 1) + 1, l_1^{\Omega_{FR}}(j_2^{(u)}))\,. \end{aligned} \tag{2.30}$$

(ii) If (α, β) is a type-1 corner point for Ω^o and $\lambda_1(\cdot)$ is nonincreasing, then for some $j_1 = 1, \ldots, n_{1,\max}^{FR}$ (2.21) holds together with

$$\begin{aligned} \frac{1}{R}x_1(0, B_1) \geq\ & x_2(l_2^{\Omega_{FR}}(j_1^{(l)}) + 1, l_2^{\Omega_{FR}}(j_1^{(l)} - 1)) \\ & -x_2(l_2^{\Omega_{FR}}(j_1^{(u)} + 1) + 1, l_2^{\Omega_{FR}}(j_1^{(u)}))\,. \end{aligned} \tag{2.31}$$

Proof of Lemma 2.25. Given a type-2 corner point (α, β), by Proposition 2.9 (i) for some $j_2 = 1, \ldots, n_{2,\max}^{FR}$ one has $\alpha = l_1^{\Omega_{FR}}(j_2) + 1$. Choosing $n = l_2^{\Omega^o}(\alpha - 1) - \beta \geq 0$, $m = \max\{(l_2^{\Omega_{FR}}(\alpha) - \beta), 0\}$, $\hat{S}^-(n) = \{l_1^{\Omega_{FR}}(j_2^{(u)} + 1) + 1, \ldots, l_1^{\Omega_{FR}}(j_2^{(u)})\} \times \{\beta, \ldots, \beta + n\} \subseteq \Omega^o$, and $\hat{S}^+(m) = \{l_1^{\Omega_{FR}}(j_2^{(l)}) + 1, \ldots, l_1^{\Omega_{FR}}(j_2^{(l)} - 1)\} \times \{\beta, \ldots, \beta + m\} \subseteq \Omega_{FR} \setminus \Omega^o$ (see Figure 2.16), it follows that the sets $\Omega^o \setminus \hat{S}^-(n)$ and $\Omega^o \cup \hat{S}^+(m)$ are CC, so $\hat{S}^-(n)$ is IR_{Ω^o} and $\hat{S}^+(m)$ is IA_{Ω^o}. By (2.15) one gets

$$J(\hat{S}^-(n))$$

$$= r_1 x_1(l_1^{\Omega_{FR}}(j_2^{(u)}+1)+1, l_1^{\Omega_{FR}}(j_2^{(u)})) + r_2 x_2(\beta, \beta+n)$$

and

$$J(\hat{S}^+(m))$$
$$= r_1 x_1(l_1^{\Omega_{FR}}(j_2^{(l)})+1, l_1^{\Omega_{FR}}(j_2^{(l)}-1)) + r_2 x_2(\beta, \beta+m)\,.$$

Combining these equalities with Lemma 2.10 (which implies $J(\hat{S}^-(n)) \geq J(\hat{S}^+(n))$), one has

$$\begin{aligned} & R(x_2(\beta,\beta+n) - x_2(\beta,\beta+m)) \\ \geq\quad & x_1(l_1^{\Omega_{FR}}(j_2^{(l)})+1, l_1^{\Omega_{FR}}(j_2^{(l)}-1)) \\ & -x_1(l_1^{\Omega_{FR}}(j_2^{(u)}+1)+1, l_1^{\Omega_{FR}}(j_2^{(u)}))\,. \end{aligned} \tag{2.32}$$

Since $0 \leq n-m \leq B_2$ (see Figure 2.16) and $\lambda_2(\cdot)$ is nonincreasing, by Lemma 2.24 one obtains

$$x_2(0, B_2) \geq x_2(\beta, \beta+n) - x_2(\beta, \beta+m)$$

which, when combined with (2.32), proves (2.30). Formula (ii) is obtained in the same way by exchanging the roles of the two classes of users. ■

Lemma 2.26. *Let $\lambda_k(\cdot)$ be nonincreasing for $k = 1, 2$.*
(i) If $\frac{1}{R} < L_1$, where

$$\begin{aligned} L_1 := & \min_{j_1=1,\ldots,n_{1,\max}^{FR}} \\ & \left\{ \frac{x_2(l_2^{\Omega_{FR}}(j_1^{(l)})+1, l_2^{\Omega_{FR}}(j_1^{(l)}-1))}{x_1(0,B_1)} \right. \\ & \left. - \frac{x_2(l_2^{\Omega_{FR}}(j_1^{(u)}+1)+1, l_2^{\Omega_{FR}}(j_1^{(u)}))}{x_1(0,B_1)} \right\}, \end{aligned}$$

then Ω^o is threshold type-1, and the threshold is equal to some $l_1^{\Omega_{FR}}(j_2)$ for some $j_2 = 0, \ldots, n_{2,\max}^{FR}$.

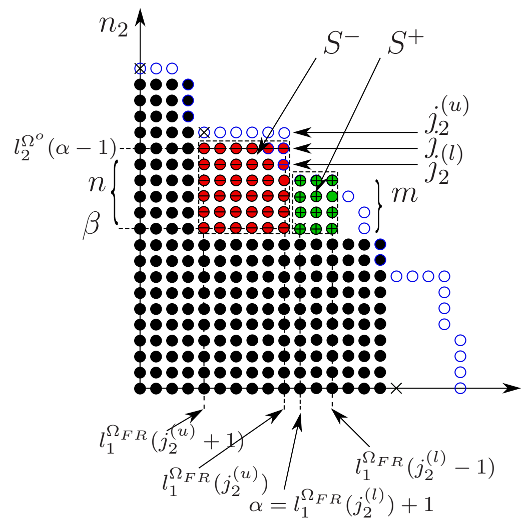


Figure 2.16: Description of a step in the proof of Lemma 2.25.

(ii) If $R < L_2$, where

$$L_2 := \min_{j_2=1,\dots,n_{2,\max}^{FR}} \left\{ \frac{x_1(l_1^{\Omega_{FR}}(j_2^{(l)})+1, l_1^{\Omega_{FR}}(j_2^{(l)}-1))}{x_2(0,B_2)} - \frac{x_1(l_1^{\Omega_{FR}}(j_2^{(u)}+1)+1, l_1^{\Omega_{FR}}(j_2^{(u)}))}{x_2(0,B_2)} \right\},$$

then Ω^o is threshold type-2, and the threshold is equal to some $l_2^{\Omega_{FR}}(j_1)$ for some $j_1 = 0, \dots, n_{1,\max}^{FR}$.

(iii) If $\frac{1}{L_1} < R < L_2$, then $\Omega^o = \Omega_{FR}$.

Proof of Lemma 2.26. (i) If $\frac{1}{R} < L_1$, then by Lemma 2.25 (ii) Ω^o has no type-1 corner points, so it is a threshold type-1 policy by Lemma 2.23. Let t_1 denote the corresponding threshold. Then either $t_1 = n_{1,\max}^{FR} := l_1^{\Omega_{FR}}(0)$ or $(t_1+1, 0)$ is a type-2 corner point for Ω^o. In the second case, by Proposition 2.9 (i) we have $t_1 + 1 = l_1^{\Omega_{FR}}(j_2) + 1$ for some $j_2 = 1, \dots, n_{2,\max}^{FR}$.

(ii) is proved similarly.

(iii) If $\frac{1}{L_1} < R < L_2$, then by parts (i) and (ii) Ω^o is both threshold type-1 and threshold type-2, so it coincides with Ω_{FR}. ■

Remark 2.27. In the particular case of a linearly-constrained feasibility region with $B_2 = 1$ (i.e., the one considered in [34]), one has $j_1^{(l)} = j_1^{(u)}$ for each $j_1 = 0, \ldots, n_{1,\max}^{FR}$, and $L_1 = \frac{1}{x_1(0,B_1)}$. So in this case Lemma 2.26 (i) reduces to [34, Theorem 1 (i)].

Proof of Theorem 2.22. For each $j_1 = 0, \ldots, n_{1,\max}^{FR}$, it follows from the definitions of $x_2(\cdot,\cdot)$ and of $j_1^{(l)}$, $j_1^{(u)}$ that

$$x_2(l_2^{\Omega_{FR}}(j_1^{(l)}) + 1, l_2^{\Omega_{FR}}(j_1^{(l)} - 1)) \geq l_2^{\Omega_{FR}}(j_1^{(l)}) + 1\,,$$

$$x_2(l_2^{\Omega_{FR}}(j_1^{(u)} + 1) + 1, l_2^{\Omega_{FR}}(j_1^{(u)})) \leq l_2^{\Omega_{FR}}(j_1^{(u)})\,,$$

and $l_2^{\Omega_{FR}}(j_1^{(u)}) = l_2^{\Omega_{FR}}(j_1^{(l)})$, so $L_1 \geq \frac{1}{x_1(0,B_1)}$. Similarly, we have $L_2 \geq \frac{1}{x_2(0,B_2)}$. ■

2.1.6 Robustness

The next proposition investigates the robustness of an optimal CC policy with respect to changes in the feasibility region, all other parameters ($\lambda_k(\cdot)$, μ_k, r_k) being unchanged.

Proposition 2.28. *Let $\Omega^o \subseteq \Omega_{FR}$ be optimal for Ω_{FR}, then for every feasibility region Ω'_{FR} such that $\Omega^o \subseteq \Omega'_{FR} \subseteq \Omega_{FR}$, Ω^o is optimal for Ω'_{FR}.*

Proof of Proposition 2.28. One has $\mathcal{P}(\Omega'_{FR}) \subseteq \mathcal{P}(\Omega_{FR})$, so $\min_{\Omega \in \mathcal{P}(\Omega'_{FR})} J(\Omega) \geq \min_{\Omega \in \mathcal{P}(\Omega_{FR})} J(\Omega)$, but one has also $\min_{\Omega \in \mathcal{P}(\Omega_{FR})} J(\Omega) = J(\Omega^o) \geq \min_{\Omega \in \mathcal{P}(\Omega'_{FR})} J(\Omega)$. ■

2.2 Generalizations to $K \geq 2$ classes

Let the number of classes $K \geq 2$ and, for a nonlinearly-constrained feasibility region, consider the problem formulation given in Section 2.1.1 with all the 2-dimensional vectors replaced by correspondent K-dimensional ones.

Definition 2.29. *A nonempty set $\Omega \subseteq \Omega_{FR} \subsetneq \mathbb{N}_0^K$ is called CC if and only if for each $\mathbf{n} \in \Omega$ with $n_k > 0$ one has $\mathbf{n} - \mathbf{e}_k \in \Omega$, where $\mathbf{e}_k$ is a K-dimensional vector whose k-th component is 1 and the other ones are 0. The CC policy associated with a CC set Ω admits an arriving request of connection if and only if after admittance the state process remains in Ω.*

Definition 2.30. *Given a K-dimensional Cartesian coordinate system, for $k \in \mathcal{K}$ and $j \in \mathbb{N}_0$ we define*

$$\mathcal{P}_k(j) = \{\mathbf{n} \in \mathbb{N}_0^K : n_k = j\}.$$

We call $\mathcal{P}_k(j)$ the (K-1)-dimensional discrete hyperplane at the j-th position along the k-th axis.

Definition 2.31. *For $S \subsetneq \mathbb{N}_0^K$ and $k = 1, \ldots, K$ we denote by $\pi_{\mathcal{K}\setminus\{k\}}(S)$ the projection of S on $\mathcal{P}_k(0)$, i.e., along the k-th axis. For a feasibility region $\Omega_{FR} \subsetneq \mathbb{N}_0^K$ and $k = 1, \ldots, K$, let $n_{k,\max}$ be the largest index j such that $\pi_{\mathcal{K}\setminus\{k\}}(\Omega_{FR} \cap \mathcal{P}_k(j)) \neq \emptyset$.*

Definition 2.32. *For a feasibility region $\Omega_{FR} \subsetneq \mathbb{N}_0^K$, the grid $G \subseteq \Omega_{FR}$ is defined as $G = \{\cap_{k=1,\ldots,K}\mathcal{P}_k : \mathcal{P}_k \in \mathcal{A}_k\} \cap (\Omega_{FR} \setminus \{0, 0, \ldots, 0\})$, where the sets $\mathcal{A}_k$ are constructed as described in the following Algorithm 2.*

The following is an informal description of Algorithm 2. Each hyperplane $\mathcal{P}_k(j)$ (Line 1, in the Algorithm 2) moves along the k-th axis starting from

Data: We take K orthogonal (K-1)-dimensional discrete hyperplanes $\mathcal{P}_k(0)$, $k = 1, \ldots, K$.

1 **foreach** $k = 1, \ldots, K$ **do**
2     $\mathcal{A}_k \leftarrow \mathcal{P}_k(0)$
3     $\mathcal{B}_k \leftarrow \pi_{\mathcal{K}\setminus\{k\}}(\Omega_{FR} \cap \mathcal{P}_k(0))$
4   **for** *($j = 1, \ldots, n_{k,\max}$)* **do**
5   **if** $\pi_{\mathcal{K}\setminus\{k\}}(\Omega_{FR} \cap \mathcal{P}_k(j)) \subsetneq \mathcal{B}_k$ **then**
6             $\mathcal{A}_k \leftarrow \mathcal{A}_k \cup \mathcal{P}_k(j)$
7             $\mathcal{B}_k \leftarrow \pi_{\mathcal{K}\setminus\{k\}}(\Omega_{FR} \cap \mathcal{P}_k(j))$
8   **end**
9   **end**
10 **end**
11 The points of G are obtained as intersections of hyperplanes in $\mathcal{A}_1, \ldots, \mathcal{A}_K$ and Ω_{FR}. The point $(0, 0, \ldots, 0)$ does not belong to the grid.

Algorithm 2: Construction of the grid G.

$j = 0$, scanning all the feasibility region Ω_{FR} (Line 4). When the cross-section $\pi_{\mathcal{K}\setminus\{k\}}(\Omega_{FR} \cap \mathcal{P}_k(j))$ changes (Line 5), the position j of the hyperplane is recorded in $\mathcal{A}_k$ (Line 6) together with the current cross-section (Line 7). The hyperplane with $j = 0$ is always considered (Line 2). The intersections among the feasibility region and the recorded discrete hyperplanes in the sets $\mathcal{A}_k$ form the points that compose the grid G (Line 11). By definition, the point $(0, 0, \ldots, 0)$ is not part of the grid.

Figure 2.17 shows a feasibility region $\Omega_{FR} \subsetneq \mathbb{N}_0^3$; for simplicity, we draw only $(\partial\Omega_{FR})^+$ and we represent it as a continuous contour. In the same way the discrete planes $\mathcal{P}_k$ are represented as continuous planes. Figure 2.17 also shows the procedure presented in Definition 2.32 - Algorithm 2.

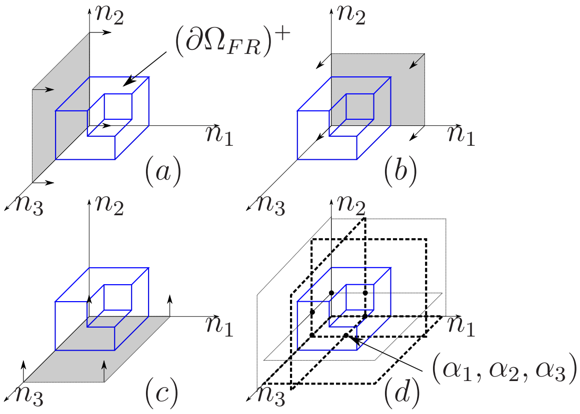


Figure 2.17: Illustration of the procedure presented in Definition 2.32 - Algorithm 2. The points of the grid are obtained as intersection of suitable hyperplanes, as shown in Figure 2.17(d) for the case of the point $(\alpha_1, \alpha_2, \alpha_3)$.

To identify a specific point $\mathbf{g} \in G$ we use the following notation. Given a grid G in a K-dimensional region, we associate each point in G with a $1 \times K$ vector $\mathbf{C}_G$, whose k-th component represents the position of the point along the k-th axis. Refer to Figure 2.18 for a graphical explanation in which the point $(2, 1)_G \in G$ in a 2-dimensional region is highlighted.

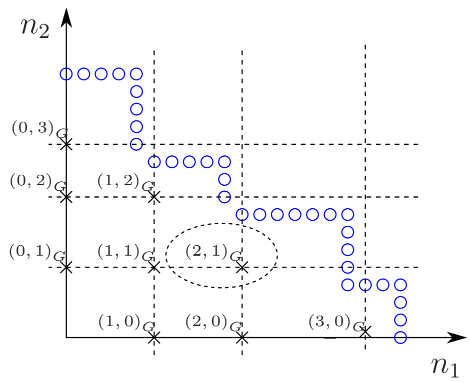


Figure 2.18: Example of enumeration of points in the grid G, with 2 classes.

Definition 2.33. *Given* $\mathbf{a}, \mathbf{b} \in G \subsetneq \mathbb{N}_0^K$, $\mathbf{b}$ *is consecutive to* $\mathbf{a}$ *with respect to* $k \in \mathcal{K}$ *if and only if*

$$\begin{cases} a_j = b_j,\ \forall j \in \mathcal{K} \setminus \{k\}; \\ b_k > a_k; \\ \nexists \mathbf{c} \in G : c_j = a_j = b_j,\ \forall j \in \mathcal{K} \setminus \{k\};\ c_k \in (a_k, b_k)\,. \end{cases}$$

Given two points $\mathbf{C}_G, (\mathbf{C}+\mathbf{e}_k)_G \in G$, with $\mathbf{e}_k$ a $1 \times K$ vector whose k-th component is 1 and the other ones are 0, it follows by Definition 2.33 that $(\mathbf{C}+\mathbf{e}_k)_G$ is consecutive to $\mathbf{C}_G$ with respect to k (see Figure 2.18). The following is an extension of Definition 2.3 to the multidimensional case. In the 2-dimensional case, in order to simplify the notation in the previous sections we denoted the second index α_2 by β.

Definition 2.34. *Given a CC policy* Ω, *the* K*-tuple* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K) \in \Omega_{FR} \setminus \Omega$ *is a corner point for* Ω *if and only if*

$$\forall k \in \mathcal{K} \text{ such that } \alpha_k \neq 0,\ \text{ one has } \boldsymbol{\alpha} - \mathbf{e}_k \in \Omega\,.$$

The following two results are extensions to the K-dimensional case of Proposition 2.9 and Theorem 2.22, respectively.

Proposition 2.35. *If the* K*-tuple* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K)$ *is a corner point of an optimal CC policy* Ω^o, *then it has to be a point of the grid* G.

Before proving Proposition 2.35, as we did before for the 2-dimensional case, we extend the definition of the objective $J(\cdot)$ to any K-dimensional region $S \subseteq \Omega_{FR}$ as follows:

$$J(S) := \frac{H(S)}{G(S)} = \frac{\sum_{\mathbf{n} \in S} (\mathbf{n} \cdot \mathbf{r}) \prod_{k=1}^{K} q_k(n_k)}{\sum_{\mathbf{n} \in S} \prod_{k=1}^{K} q_k(n_k)}\,. \tag{2.33}$$

Note that, for any CC Ω and $S_1 \subseteq \Omega \subseteq S_2$ with S_1 and S_2 hyper-rectangles, it follows from formula (2.33) that $J(S_2) \geq J(\Omega) \geq J(S_1)$ (instead, this is not true in general when S_1 and S_2 are not hyper-rectangles). Moreover, given a K-dimensional region S of the form

$$S := \Big\{ \mathbf{n} \in \Omega_{FR} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S' \text{ and } \mathbf{n}^{(2)} \in \{a\mathbf{e}_k, \ldots, b\mathbf{e}_k\} \Big\}, \tag{2.34}$$

where $a, b \in \mathbb{N}_0, a \leq b$ and $S' \subsetneq N_0^K$ is made up of points whose k-th component is 0, it follows from formula (2.33) that

$$J(S) = J(S') + r_k x_k(a, b) \,. \tag{2.35}$$

Proof of Proposition 2.35. We show that if a corner point $\boldsymbol{\alpha} \notin G$ exists for an optimal CC policy Ω^o, then we can construct 2 CC regions $S^- \subsetneq \Omega_{FR}, S^+ \subsetneq \Omega_{FR}$ such that a necessary condition for the optimality of Ω^o is violated.

By construction of the grid G (Definition 2.32), since $\boldsymbol{\alpha} \notin G$ there exists an index $k \in \mathcal{K}$ such that the component α_k of $\boldsymbol{\alpha}$ is not in the set of possible values assumed by the k-th coordinate of a point of the grid. Let $\hat{\alpha}_k$ be the largest value smaller than α_k that can be assumed by the k-th coordinate of a point of the grid, and $\hat{\mathbf{g}}$ a point of the grid whose k-th component is $\hat{\alpha}_k$ and whose associated vector is $\hat{\mathbf{C}}_G$. Without loss of generality, we suppose that there is no other corner point $\tilde{\boldsymbol{\alpha}}$ of Ω^o whose k-th component $\tilde{\alpha}_k$ satisfies $\hat{\alpha}_k < \tilde{\alpha}_k < \alpha_k$. Let

$$S'_{\hat{\alpha}_k} := \Omega \cap \mathcal{P}_k(\hat{\alpha}_k) \,,$$

$$S'_{\alpha_k} := \Omega \cap \mathcal{P}_k(\alpha_k)$$

and

$$S' := \pi_{\mathcal{K} \setminus \{k\}}(S'_{\hat{\alpha}_k}) \setminus \pi_{\mathcal{K} \setminus \{k\}}(S'_{\alpha_k}) \,.$$

Then, the set

$$S^- := \Big\{ \mathbf{n} \in \Omega_{FR} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S' \quad \text{and}\, \mathbf{n}^{(2)} = (\alpha_k - 1)\mathbf{e}_k \Big\} \tag{2.36}$$

is IR_{Ω^o}, whereas, by the construction of the grid, it follows that between $\hat{\mathbf{C}}_G$ and its consecutive point $(\hat{\mathbf{C}} + \mathbf{e})_G$ the cross-section of Ω_{FR} along the k-th axis does not change, so

$$S^+ := \Big\{ \mathbf{n} \in \Omega_{FR} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S' \quad \text{and}\, \mathbf{n}^{(2)} = \alpha_k \mathbf{e}_k \Big\} \tag{2.37}$$

is a subset of Ω_{FR} and is IA_{Ω^o}. By formula (2.35), we get

$$\begin{aligned} J(S^-) =& J(S') + r_k(\alpha_k - 1)\,, \\ J(S^+) =& J(S') + r_k \alpha_k\,. \end{aligned}$$

So one gets $J(S^-) < J(S^+)$, which contradicts the optimality condition stated in Lemma 2.10, and one concludes that if a corner point of Ω^o exists, then it has to be in the grid G. ■

Theorem 2.36. *Let $\lambda_k(\cdot)$ be nonincreasing for $k = 1, \ldots, K$. The following holds*

(i) If for a given $k \in \mathcal{K}$ and for all $\bar{\alpha}_k$ that are k-th coordinates of points of the grid G, one has

$$r_k > \frac{\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max})}{\bar{\alpha}_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1)}\,. \tag{2.38}$$

then all the possible corner points of Ω^o have the k-th coordinate $\alpha_k = 0$.

(ii) If condition (2.38) holds for all $k \in \mathcal{K}$ and for all $\bar{\alpha}_k$ that are k-th coordinates of points of the grid G, then $\Omega^o = \Omega_{FR}$.

The proof of Theorem 2.36 is based on the following lemma, which is an extension of Lemma 2.26 to the multidimensional case.

Lemma 2.37. *If the K-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_K)$ is a corner point for Ω^o, and $\lambda_1(\cdot), \lambda_2(\cdot), \dots, \lambda_K(\cdot)$ are not increasing, then, for all k such that $\alpha_k \neq 0$, one has*

$$\begin{aligned} &\textstyle\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max}) \\ &\geq \textstyle\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max} - \alpha_j) \\ &\geq r_k \left(\alpha_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1)\right) \quad . \end{aligned} \tag{2.39}$$

Proof of Lemma 2.37. Let $\Omega^o \subseteq \Omega_{FR} \subsetneq \mathbb{N}_0^K$ be an optimal CC policy, and the K-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_K)$ be a corner point of Ω^o. From Proposition 2.35 we know that $\boldsymbol{\alpha} \in G$. For each of its components $\alpha_k \neq 0$ we can construct two CC regions S_k^- and S_k^+ as follows. Likewise in the proof of Proposition 2.35, let $\hat{\alpha}_k$ be the largest value smaller than α_k that can be assumed by the k-th coordinate of a point of the grid, and $\hat{\mathbf{g}}$ a point of the grid whose k-th component is $\hat{\alpha}_k$ and whose associated vector is $\hat{\mathbf{C}}_G$. Let

$$S'_{\hat{\alpha}_k} := \Omega \cap \mathcal{P}_k(\hat{\alpha}_k)\,,$$

$$S'_{\alpha_k} := \Omega \cap \mathcal{P}_k(\alpha_k)\,,$$

$$\Omega'_{FR,\alpha_k} := \Omega_{FR} \cap \mathcal{P}_k(\alpha_k)$$

$$S_k'^- := \pi_{\mathcal{K} \setminus \{k\}}(S'_{\hat{\alpha}_k}) \setminus \pi_{\mathcal{K} \setminus \{k\}}(S'_{\alpha_k})\,,$$

and

$$S_k'^+ := S_k'^- \cap \pi_{\mathcal{K} \setminus \{k\}}(\Omega'_{FR,\alpha_k})\,.$$

Then, the set

$$S_k^- := \Big\{ \mathbf{n} \in \Omega_{FR} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S_k'^- \text{ and } \mathbf{n}^{(2)} \in \{\hat{\alpha}_k \mathbf{e}_k, \dots, (\alpha_k - 1)\mathbf{e}_k\} \Big\} \tag{2.40}$$

is clearly IR_{Ω^o}, whereas

$$S_k^+ := \Big\{ \mathbf{n} \in \Omega_{FR} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S_k'^+ \text{ and } \mathbf{n}^{(2)} = \alpha_k \mathbf{e}_k \Big\} \tag{2.41}$$

is a subset of Ω_{FR} and is also IA_{Ω^o}. Using formula (2.35), we obtain:

$$J(S_k^-) = J(S_k'^-) + r_k x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1)$$

and

$$\begin{aligned} J(S_k^+) &= J(S_k'^+) + r_k x_k(\mathbf{C}_{G,k}, \mathbf{C}_{G,k}) \\ &= J(S_k'^+) + \alpha_k r_k \,. \end{aligned}$$

Combining these equalities with Lemma 2.10, we have

$$\begin{aligned} J(S_k^-) &= J(S_k'^-) + r_k x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1) \\ &\geq J(S_k'^+) + \alpha_k r_k = J(S_k^+) \,. \end{aligned}$$

Now, $J(S_k'^+)$ is bounded from below by 0, whereas $J(S_k'^-)$ is bounded from above by $J(S_k'^{ub})$, where $S_k'^{ub}$ is any (K-1)-dimensional hyper-rectangle that contains $S_k'^-$, for instance, the hyper-rectangle $\pi_{\mathcal{K}\setminus\{k\}} \left(\Pi_{j=1}^{K} \{\alpha_j, \ldots, n_{j,\max}\} \right)$. So, with this choice of $S_k'^{ub}$, one gets

$$\begin{aligned} J(S_k'^{ub}) &= \sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j(\alpha_j, n_{j,\max}) \\ &\geq r_k \left(\alpha_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1) \right) \,. \end{aligned}$$

Since $\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_K(\cdot)$ are not increasing, by Lemma 2.24 we get

$$\begin{aligned} &\textstyle\sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j(0, n_{j,\max} - \alpha_j) \\ &\geq r_k \left(\alpha_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1) \right) \,, \end{aligned}$$

whereas

$$\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max}) \geq \sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max} - \alpha_j)$$

follows by the expression of x_j in (2.27). ■

Proof of Theorem 2.36. (i) If a corner point of Ω^o has its k-th coordinate α_k different from 0, then by Proposition 2.35 α_k is equal to the k-th coordinate $\bar{\alpha}_k$ of some point in the grid G. Moreover, by Lemma 2.37, the condition (2.39) must hold. This is in contradiction with the assumption (2.38), so one has $\alpha_k = 0$.
(ii) Proceeding likewise in the proof of (i), it follows that all the potential corner points of Ω^o have $\alpha_k = 0$, for all $k \in \mathcal{K}$. Since $(0, 0, \ldots, 0)$ is never a corner point, it follows that Ω^o has no corner points, so it is the complete sharing policy.

■

Propositions 2.7 and 2.12 can be extended to the case of $K \geq 2$ with similar proofs based on cross-section arguments and an induction argument with respect to the number of classes.

2.3 Simulation results

2.3.1 Criteria to improve suboptimal CC policies

To verify the effectiveness of the criteria proposed in Propositions 2.4 and 2.5, we provide some simulation results. We consider a cell in a cellular network and two classes of traffic, i.e., voice call traffic (class 1) and data traffic (class 2), modelled by Poisson arrivals and exponential call durations (as often done in the literature; see, e.g., [51]). For the class-1 traffic we have (on average) 20 calls per time unit, e.g., per minute ($\lambda_1 = 20$), with an average holding time of 3 time units per call ($\mu_1 = 1/3$). For class-2 traffic we set $\lambda_2 = 10$ and $\mu_2 = 1/20$. The per-call instantaneous revenues of the two classes are the same ($r_1 = r_2 = 1$). The Ω_{FR} used in the simulations has a nonlinear upper boundary that models QoS constraints, like the ones in [9, Figure 3] and [1, pp. 46-49].

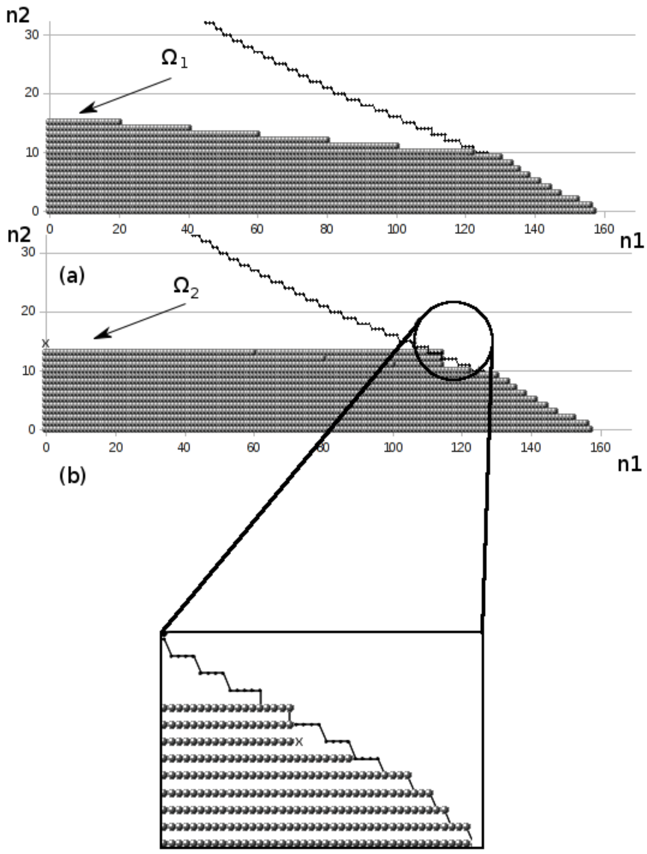


Figure 2.19: (a) initial CC policy Ω_1; (b) final CC policy Ω_2.

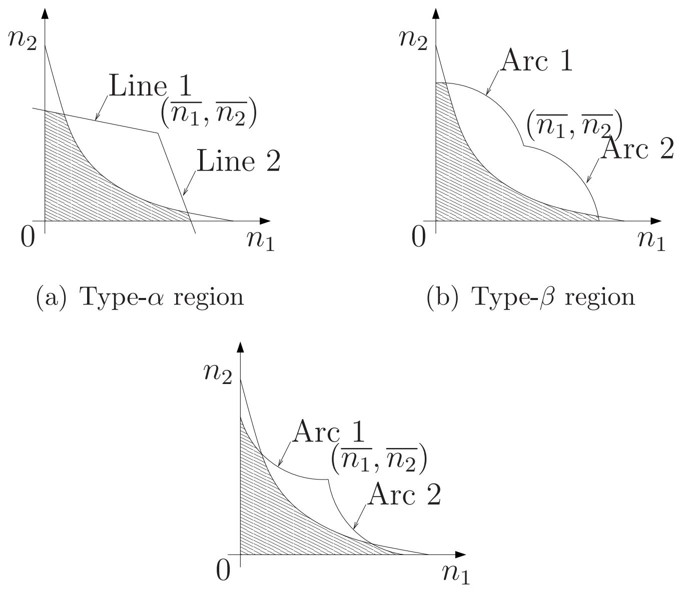
Starting from the initial CC policy Ω_1 depicted in Figure 2.19(a), the final CC policy Ω_2 in Figure 2.19(b) has been obtained by applying Proposition 2.4 four times, to suitable corner points. The initial value of the objective is $J(\Omega_1) = 66.4229$, whereas the final value is $J(\Omega_2) = 73.9256$, with an improvement of 11.3%. Note from Figure 2.19(b) that the CC policy Ω_2 cannot be further improved via Propositions 2.4 or 2.5 and that its corner points have the structural properties stated in Proposition 2.8 and Theorem 2.11.

2.3.2 Optimality of threshold-type policies

To show the shape of the optimal CC policies of threshold type, in this section we provide some simulation results under the assumptions of Theorem 2.22. More precisely, we have considered different kinds of CC policies of special structure and dependent on a small number of parameters in the simulation campaigns and we have used them in a heuristic search method to find the optimal CC policy. Moreover, using the knowledge from Theorem 2.22 about the optimal policies, we have found with a simple search algorithm the optimal threshold type policy.

Let $K = 2$; Figure 2.20 shows the CC sets associated to the CC policies considered in the simulations.

Given the feasibility region Ω_{FR} reported in Figure 2.20, the region Ω shown in Figure 2.20(a) depends on four parameters: $\overline{n_1}$, $\overline{n_2}$, m_1, m_2. $\overline{n_1}$ and $\overline{n_2}$ represent the coordinates of a generic point in the first (n_1, n_2) quadrant $(n_1 \geq 0,\, n_2 \geq 0)$, and outside the feasibility region. m_1 and m_2 are the slopes of the two lines (Line 1 and Line 2) shown in Figure 2.20(a), originating from the point $(\overline{n_1}, \overline{n_2})$ and cutting the feasibility region. Formula (2.42) describes Lines 1 and 2. Region Ω, derived from the intersection of Ω_{FR} and the area between the two lines (Lines 1 and 2) and the two axes, is defined as a type-α region.



(a) Type-α region (b) Type-β region

(c) Type-γ region

Figure 2.20: The shapes of the CC sets considered in the simulations in Section 2.3.2.

$$\begin{cases} n_2(n_1) = m_1(\overline{n_1} - n_1) + \overline{n_2} & \text{Line 1} \\ n_2(n_1) = \overline{n_2} - \frac{n_1 - \overline{n_1}}{m_2} & \text{Line 2} \\ m_1 \geq 0, m_2 \geq 0\,. \end{cases} \tag{2.42}$$

The other two kinds of regions considered in the simulations are obtained by cutting the feasibility region with parabolic arcs. Each of them is described by four parameters: $\overline{n_1}$, $\overline{n_2}$, m_1, and m_2. The parameters $\overline{n_1}$, $\overline{n_2}$ have the same meanings as before, whereas m_1 and m_2 refer to the degrees of concavity/convexity of the arcs of parabolas. Formulas (2.43) and (2.44) describe such arcs. Region Ω called type-β, in Figure 2.20(b), is derived from the intersection of Ω_{FR}, the area between the two arcs in (2.43 and, the two axes. Region Ω called type-γ, in Figure 2.20(c), is obtained in the same way but using the arcs in (2.44).

$$\begin{cases} n_2(n_1) = \sqrt{\frac{\overline{n_1}-n_1}{m_1}} + \overline{n_2} & \text{Arc 1} \\ n_2(n_1) = \overline{n_2} - m_2(n_1 - \overline{n_1})^2 & \text{Arc 2} \\ m_1 \geq 0, m_2 \geq 0\,. \end{cases} \quad (2.43)$$

$$\begin{cases} n_2(n_1) = m_1(n_1 - \overline{n_1})^2 + \overline{n_2} & \text{Arc' 1} \\ n_2(n_1) = \overline{n_2} - \sqrt{\frac{n_1-\overline{n_1}}{m_2}} & \text{Arc' 2} \\ m_1 \geq 0, m_2 \geq 0\,. \end{cases} \quad (2.44)$$

The simulation campaigns were carried on as follows. We fixed the traffic parameters and the feasibility region Ω_{FR}. For each of the three shapes of CC sets shown in Figure 2.20, the optimal parameters $(\overline{n_1}^o, \overline{n_2}^o, m_1^o, m_2^o)$ that maximize $J(\cdot)$ were searched by using the Nelder-Mead algorithm [52, 53], which is a direct search algorithm for non-linear optimization [54].

In the first campaign we set the simulation setup to fit condition (i) in Theorem 2.22. We assumed homogeneous Poisson arrivals for each class (i.e., $\lambda_k(j)$ do not depend on j, so they were replaced by λ_k). We considered the feasibility region Ω_{FR} taken from [1, pp. 46-49] and reported in Figure 2.21, for which one has $n_{1,\max}^{FR} = 159$, $n_{2,\max}^{FR} = 59$, $B_1 = 4$, and $B_2 = 5$. Then we made the choices $r_1 = \mu_1/(\lambda_1+\lambda_2)$, and $r_2 = \mu_2/(\lambda_1+\lambda_2)$. The choices have been performed to conform Proposition 2.2 and to use the Equation (2.5). With $\rho_1 := \frac{\lambda_1}{\mu_1} = 100$, $\rho_2 = 30$, $\lambda_1 = 50$, $\lambda_2 = 90$, $\mu_1 = 0.5$, and $\mu_2 = 3$, we got $R = \mu_2/\mu_1 = 6$ and $x_1(0,4) \simeq 3.9$. Then $R > x_1(0, B_1)$ and by Theorem 2.22 (i) the optimal CC policy is threshold type-1.

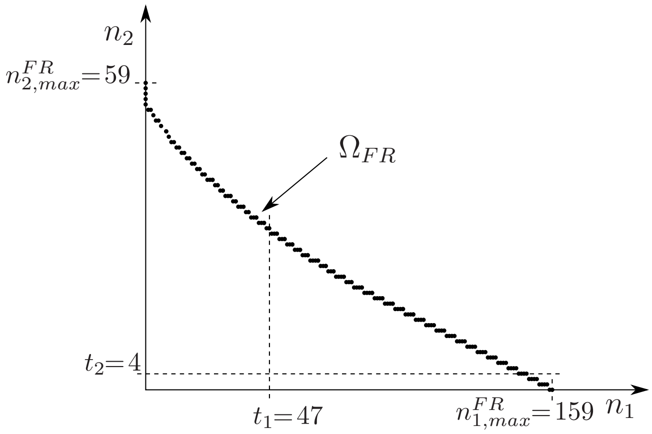


Figure 2.21: The feasibility region used during simulations in Section 2.3.2 and the optimal threshold type-1 policy.

The best policies found by the Nelder-Mead algorithm are shown in Table 2.2 for each kind of subregion.

Region form	$\overline{n_1}^o$	$\overline{n_2}^o$	m_1^o	m_2^o	$J(\Omega)$	Optimal threshold t_1
Type-α	47	32	4.41	0.03	0.7468	-
Type-β	47	66	33.35	77.66	0.7468	-
Type-γ	47	68	0.08	0.00	0.7468	-
Th. type-1	-	-	-	-	0.7468	47

Table 2.2: Simulation results for the case of an optimal CC policy of threshold type-1.

Note that the best type-α policy found by the Nelder-Mead algorithm has the following values for the parameters: $\overline{n_1}^o = 47$, $\overline{n_2}^o = 32$, $m_1^o = 4.41$, and $m_2^o = 0.03$ shown in the first line in Table 2.2. This means that the point $(\overline{n_1}^o, \overline{n_2}^o)$ is outside the feasibility region, Line 2 is almost vertical, and Line

1 does not intersect the feasibility region. The same considerations hold true for the best type-β and type-γ policies found by the Nelder-Mead algorithm and shown in Table 2.2 (second and third line). Due to the discretization of the call space, the three 4-tuples of parameters for the best type-α, type-β, and type-γ policies shown in Table 2.2 describe the same CC set, so they are associated with the same threshold type-1 policy (this is the reason for which the column $J(\Omega)$ in Table 2.2 has identical entries).

The optimal region in the last row has been obtained using the knowledge of Theorem 2.22 that assures us that the threshold type-1 policy is the optimal one. It is important to note that Theorem 2.22 give us all the possible positions of the type-1 threshold t_1 ($l_1^{\Omega_{FR}}(j_2), j_2 = 0, \ldots, n_{2,\max}^{FR}$). A simple search algorithm has been used to compute all the possible positions (59) and to find the optimal one. This value is (59) negligible compared to the computational complexity of the Nelder-Mead algorithm (3 orders of magnitude bigger) used to find the optimal policies from Type-α, Type-β, and Type-γ regions.

For the second campaign of simulations, we have set the simulation parameters to satisfy the conditions in Theorem 2.22 (ii). Referring to the objective (2.5) with $r_1 = \mu_1/(\lambda_1+\lambda_2)$ and $r_2 = \mu_2/(\lambda_1+\lambda_2)$, by exploiting Proposition 2.2 and choosing $\rho_1 = 100$, $\rho_2 = 30$, $\lambda_1 = 500$, $\lambda_2 = 30$, $\mu_1 = 5$, and $\mu_2 = 1$, one gets $R = \mu_2/\mu_1 = 1/5$ and $x_2(0,5) \simeq 4.8$. Then $1/R > x_2(0, B_2)$ and for Theorem 2.22 (ii) the optimal CC policy is threshold type-2. The optimal type-2 threshold region and the best policies found by the Nelder-Mead algorithm are shown in Table 2.3 (for each kind of subregion). Comments are similar to the case in Table 2.2.

Region form	$\overline{n_1}^o$	$\overline{n_2}^o$	m_1^o	m_2^o	$J(\Omega)$	Optimal threshold t_2
Type-α	141	4	0.00	1.08	0.9947	-
Type-β	216	3	76.42	4.31	0.9947	-
Type-γ	160	4	0	0.03	0.9947	-
Th. type-2	-	-	-	-	0.9947	4

Table 2.3: Simulation results for the case of an optimal CC policy of threshold type-2.

2.3.3 Optimal thresholds

In the following, we investigate the position of the optimal threshold under the conditions of Theorem 2.22. More precisely, we verify that, under the conditions of Theorem 2.22 (i) and (ii) resp., the optimal threshold for threshold type-1 policies is equal to $l_1^{\Omega_{FR}}(j)$ for some $j = 0, \ldots, n_{2,\max}^{FR}$, and the optimal threshold for threshold type-2 policies is equal to $l_2^{\Omega_{FR}}(j)$ for some $j = 0, \ldots, n_{1,\max}^{FR}$. The feasibility region used to make these simulations is shown in Figure 2.22. We assume homogeneous Poisson arrivals for both classes.

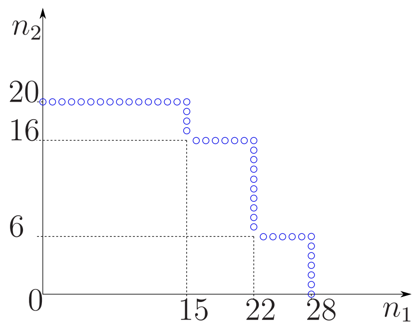


Figure 2.22: The feasibility region considered in Section 2.3.3.

With this feasibility region, one has $B_1 = 16$ and $B_2 = 10$; for $\lambda_1 = 50$, $\lambda_2 = 150$, $\mu_1 = 0.5$, $\mu_2 = 5$, $r_1 = 0.25$, and $r_2 = 2.5$ one gets $R = r_2/r_1 = 10$ and $x_1(0, 10) \simeq 9.89$. Then $R > x_1(0, B_1)$ and by Theorem 2.22 (i) there exists an optimal CC policy of threshold type-1. According to Theorem 2.22 (i), the optimal threshold belongs to the set $\{0, 15, 22, 28\}$. Table 2.4 shows that this is indeed the case and that the optimal threshold is $t_1 = 15$.

Threshold t_1	$J(\Omega)$
15	50.2007
14	49.9540
13	49.7072
12	49.4604
11	49.2135
10	48.9665
16	47.0714
17	43.5764
20	42.6669
18	42.4987
19	42.4718

Table 2.4: Simulation results for the case of an optimal CC policy of threshold type-1.

Threshold t_2	$J(\Omega)$
6	28.035315
5	28.033691
4	28.032067
3	28.030443
7	28.030139
8	28.010752
9	27.947111
10	27.764107
11	27.318421
12	26.459426
13	25.255737

Table 2.5: Simulation results for the case of an optimal CC policy that of threshold type-2.

For $\lambda_1 = 200$, $\lambda_2 = 3$, $\mu_1 = 2$, $\mu_2 = 0.1$, $r_1 = 1$, and $r_2 = 0.05$ one has $R = r_2/r_1 = 0.05$ and $x_2(0, 16) \simeq 15.08$. Then one obtains $1/R > x_2(0, B_2)$ and by Theorem 2.22 (ii) there exists an optimal CC policy of threshold type-2. According to Theorem 2.22 (ii), the optimal threshold belongs to the set $\{6, 16, 20\}$. Table 2.5 shows that this is indeed the case and that the optimal threshold is $t_2 = 6$.

2.3.4 Robustness

Finally, we considered the effect of changing the feasibility region itself, all other parameters (λ_k, μ_k, r_k) being unchanged, as before, we assume homogeneous Poisson arrivals.

With the feasibility region Ω_{FR} shown in Figure 2.23 and the parameter values $\lambda_1 = 50$, $\lambda_2 = 90$, $\mu_1 = 0.5$, $\mu_2 = 3$, $r_1 = 1$, and $r_2 = 6$, by the same technique applied in Section 2.3.3 it follows that the optimal CC policy Ω^o is threshold type-1 with threshold $t_1 = 47$, as shown in Figure 2.23. Then, we have modified the original feasibility region Ω_{FR} by generating three other feasibility regions that are obtained from the original one by removing states that do not belong to Ω^o (Ω_{FR} Mod 1, Ω_{FR} Mod 2, Ω_{FR} Mod 3 in Figure 2.23). So, the assumptions of Proposition 2.28 about the robustness of the optimal CC policies with respect to changes in the feasibility region are satisfied, and Ω^o is still optimal for each of these three modified feasibility regions.

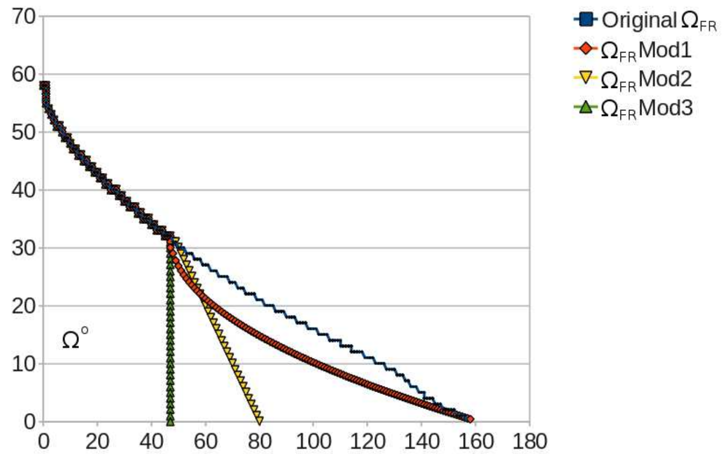


Figure 2.23: The original feasibility region and three modified feasibility regions obtained by removing states that do not belong to the CC set associated with an optimal CC policy Ω^o.

Chapter 3

Conclusions

3.1 Comparisons with previous results

In [9, Section IV-A and Section IV-C], for a similar[1] CAC model with nonlinearly-constrained feasibility regions, the authors find a sufficient condition for which the optimal policy is complete sharing (called *greedy* policy in [9] and *double-threshold* policy in [34]). For $K = 2$, the sufficient condition for the optimality of complete sharing given in [9, formula 34] is (rewritten using our notation)

$$\rho_1 := \frac{\lambda_1}{\mu_1} \leq R \leq \frac{\mu_2}{\lambda_2} := \frac{1}{\rho_2} \,. \tag{3.1}$$

The sufficient condition for the optimality of complete sharing, which follows

[1]The only significant difference between the two models is in the discretizations of the time variable in [9] and consequently of the arrival and departure processes. Our model can be considered as the limit of the one in [9] when the sampling interval tends to 0. A second minor difference is that the optimization problem in [9, Section IV-A] has a finite horizon; however, in [9, Section IV-C] the authors show that their solution solves also a discrete-time version of the infinite-horizon optimization problem with average reward per-unit time defined by the objective (2.1).

by our Theorem 2.22 is

$$\frac{1}{L_1} \leq x_1(0, B_1) < R < \frac{1}{x_2(0, B_2)} \leq L_2 \,. \tag{3.2}$$

The following relationships hold.

Proposition 3.1.

$$\frac{1}{L_1} \leq x_1(0, B_1) < \rho_1 < R < \frac{1}{\rho_2} < \frac{1}{x_2(0, B_2)} \leq L_2 \,.$$

Proof of Proposition 3.1. From the definition of the function $x_k(\cdot, \cdot)$, one obtains

$$\begin{aligned} x_1(0, B_1) &= \frac{\sum_{j=0}^{B_1} j \frac{\rho_1^j}{j!}}{\sum_{j=0}^{B_1} \frac{\rho_1^j}{j!}} = \frac{\rho_1 \sum_{j=1}^{B_1} \frac{\rho_1^{j-1}}{(j-1)!}}{\sum_{j=0}^{B_1} \frac{\rho_1^j}{j!}} = \\ &= \frac{\rho_1 \sum_{j=1}^{B_1 - 1} \frac{\rho_1^j}{j!}}{\sum_{j=0}^{B_1} \frac{\rho_1^j}{j!}} \,. \end{aligned} \tag{3.3}$$

As $\frac{\sum_{j=1}^{B_1-1} \frac{\rho_1^j}{j!}}{\sum_{j=0}^{B_1} \frac{\rho_1^j}{j!}}$ is smaller than 1, one has

$$x_1(0, B_1) < \rho_1 \,.$$

The following result is obtained in the same way:

$$x_2(0, B_2) < \rho_2 \,.$$

Finally, one has

$$\frac{1}{L_1} \leq x_1(0, B_1) < \rho_1 < R < \frac{1}{\rho_2} < \frac{1}{x_2(0, B_2)} \leq L_2 \,.$$

■

Proposition 3.1 implies that the sufficient condition (3.2) for the optimality of complete sharing is less restrictive than condition (3.1).

In [9, Section IV-C], the following sufficient condition for the optimality of complete sharing for a feasibility region with $K > 2$ classes is given:

$$r_k \geq \sum_{j \in \mathcal{K} \setminus \{k\}} r_j \frac{\lambda_j}{\mu_j}, \ \forall k \in \mathcal{K} \,. \tag{3.4}$$

On the other hand, the sufficient condition from our Theorem 2.36 is

$$r_k > \frac{\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max})}{\bar{\alpha}_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1)} \,, \tag{3.5}$$

for all $k \in \mathcal{K}$ and all $\bar{\alpha}_k$ that are k-th coordinates of points of the grid G.

Proposition 3.2. *For all $k \in \mathcal{K}$ and all $\bar{\alpha}_k$ that are k-th coordinates of points of the grid G, one has*

$$\frac{\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max})}{\bar{\alpha}_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1)} < \sum_{j \in \mathcal{K} \setminus \{k\}} r_j \frac{\lambda_j}{\mu_j} \,.$$

Proof of Proposition 3.2. Since $\bar{\alpha}_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1) \geq 1$, we have

$$\frac{\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max})}{\bar{\alpha}_k - x_k((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1)} \leq \sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max}) \,.$$

Moreover, proceeding likewise in formula (3.3), one obtains

$$\sum_{j \in \mathcal{K} \setminus \{k\}} r_j x_j(0, n_{j,\max}) < \sum_{j \in \mathcal{K} \setminus \{k\}} r_j \rho_j = \sum_{j \in \mathcal{K} \setminus \{k\}} r_j \frac{\lambda_j}{\mu_j} \,,$$

which concludes the proof.

■

Proposition 3.2 implies that the sufficient conditions expressed in (3.4) are more restrictive than ours in (3.5).

3.2 Final discussions

Call Admission Control is a tool to guarantee Quality of Service. CAC can be used in every network application where it is possible to define the concept of connection. Phone calls as well as (focusing on IP-based traffic) VoIP and WEB connections can benefit from CAC. Poisson arrivals and exponential call durations are often assumed in the literature (see e.g. [51]). CAC problems with nonlinearly-constrained feasibility regions arise, e.g., in the context of wireless networks [9].

In this thesis we are derived optimality conditions for Call Admission Control problems with nonlinearly-constrained feasibility regions and K classes of users. Call admission strategies are restricted to the family of coordinate-convex policies. For two classes of users, different contributions about: general structural properties, number of optimal policies, structural properties depending on the revenue ratio and, robustness to perturbations in the feasibility region boundaries are introduced. Then, the analysis is generalized to the case of $K \geq 2$ classes of users. The theoretical results are exploited for narrowing the set of coordinate-convex admissible policies. Up to our knowledge, the problem of deriving structural properties of the optimal CC policies in the case of general nonlinearly-constrained feasibility regions has received little attention till now.

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