NP-completeness Proof of OCLB Problem

BalCon: A Distributed Elastic SDN Control via Efficient Switch Migration

Marco Cello*, Yang Xu†, Anwar Walid‡, Gordon Wilfong‡, H. Jonathan Chao† and Mario Marchese§

* *Nokia Bell Labs, Dublin, Ireland. Email: marco.cello@nokia-bell-labs.com*

†*NYU Tandon School of Engineering, New York, NY, USA. Email: yang@nyu.edu, chao@nyu.edu*

‡*Nokia Bell Labs, Murray Hill, NJ, USA. Email:anwar.walid@nokia-bell-labs.com, gordon.wilfong@nokia-bell-labs.com*

§*University of Genoa, Genoa, Italy. Email:mario.marchese@unige.it*

I. NP-COMPLETENESS PROOF

Definitions: In what follows we have a network $H = (W, A)$ where the nodes in W represent routers and arcs in A are symmetric (i.e., $(u, v) \in A$ if and only if $(v, u) \in A$. A flow f in a network H is a directed acyclic path in H. Consider a partition of W into regions $R_1, R_2, \ldots, R_c$. For a flow f we cut f into maximal sections contained in a region $f_1, f_2, \ldots, f_t$. That is, f_i and f_{i+1} are contained regions R_a and R_b respectively where $R_a \neq R_b$. We then say that f *starts* at the first router (node) in each f_i.

Given a set of flows F, a bound B on the number of flows that can start at a router and a bound K on the total number of flows starting at routers within any given region, we say that a partition of W is *valid* if no router or controller bound is exceeded.

Lower Bounds: We consider the following version of the problem concerning multiple controllers and show that it is NP-complete.

Given a network $H = (W, A)$, c the number of available controllers, a collection F of flows (directed paths in H), a bound B on the number of flows starting at any router $w \in W$, and a bound K on the total number of flows starting at routers controlled by a given controller, we ask the question as to whether there exists a valid partitioning of W amongst the c controllers is NP-complete. We call this problem the VALID PARTITIONING problem.

Theorem 1.1: The VALID PARTITIONING problem is NP-complete.

Proof: Given a partitioning it is easy to verify that it is valid in polynomial time and so VALID PARTITIONING is in NP.

Consider the MINIMUM CUT INTO EQUAL-SIZED SUBSETS problem where one is given an unweighted graph $G = (V, E)$, two vertices s and t, a bound d and the question is whether there exists a partitioning of V into two equal parts V_s and V_t where $s \in V_s$, $t \in V_t$ so that at most d edges cross between V_t and V_s. The MINIMUM CUT INTO EQUAL-SIZED SUBSETS problem is known to be NP-complete [1].

We show that VALID PARTITIONING is NP-hard using a reduction from MINIMUM CUT INTO EQUAL-SIZED SUBSETS. The basic idea of the construction is that we build a network $H = (V, A)$ with a cycle C_v for each node in $v \in V$ where for each edge $e = uv$ in E there are arcs (u_i, v_j) and (v_j, u_i) in A. See Figure 1 for an example construction of H given G. There will be flows to ensure that each cycle must be completely contained in a region, C_s and C_t must be in different regions and at most n other cycles can be in a region for otherwise the bound on the number of flows starting in a region will be exceeded. Each arc pair $(u_i, v_j), (v_j, u_i)$ where C_u and C_v are in different regions will give rise to an additional flow starting at u_i and an additional flow starting at v_j. We use this fact to ensure that the bound on the number of flows starting at a node in a given region is not exceeded if and only if there are at most d such node pairs.

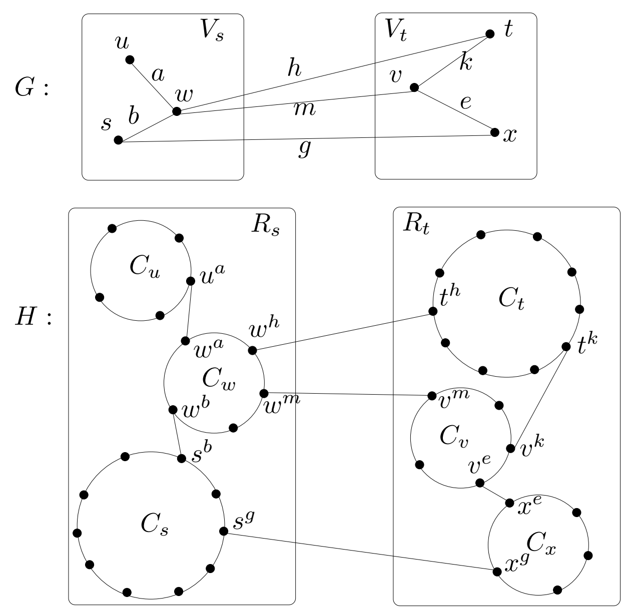

Figure 1. NP-hardness Construction.

Suppose we are given an instance I of MINIMUM CUT INTO EQUAL-SIZED SUBSETS as above where $|V| = 2n$. Let $\Delta > 1$ be the maximum degree of any node in G and choose $D > \Delta + d$. Let M be some integer greater than $d + 1$.

We now define an instance P of VALID PARTITIONING on a graph $H = (W, A)$ as follows.

At most $B = M + 1$ flows are allowed to start at each router. The number of available controllers c is 2.

Choose $Q > nD + d$. We define the bound on the total number of flows starting at routers in a region as

$$K = M(Q + nD) + d. \quad (1)$$

Then note that

$$2QM > K \text{ (since } Q > nD + d) \quad (2)$$

and

$$M(Q + (n + 1)D) > K \text{ (since } D > d). \quad (3)$$

For each node $v \neq s, t$ in V there is bidirectional cycle C_v of nodes $v_0, v_1, \ldots, v_{D-1}$ in H. For nodes s and t, there are bidirectional cycles C_s and C_t in H with nodes $s_0, s_1, \ldots, s_{Q-1}$ and $t_0, t_1, \ldots, t_{Q-1}$ respectively.

For each edge $e = uv \in E$ choose some $0 \leq i, j < D$ and define $v^e = v_i$ and $u^e = u_j$ so that for all $f = uw, g = vx$ where $e \neq f, g$, then $v^f \neq v^e$ and $u^g \neq u^e$. That is, for each node $v \in V$ we assign a distinct v_i to each edge in E having v as an endpoint.

For each node $v \neq s, t$ in V for each v_i in C_v, if $v_i = v^e$ for some $e = vw \in E$ then there are $M - 1$ flows $v_{i-1}v_i$ and one flow $v_{i-1}v^e w^e$. If $v_i \neq v^e$ for any e then there are M flows $v_{i-1}v_i$. If $v \neq s, t$ then indices are computed $\bmod D$ and if $v = s, t$ then indices are computed $\bmod Q$.

Notice that if an edge $v_{i-1}v_i$ is in a cut then $2M > M + 1 = B$ flows will start at v_i. Thus in a valid partition each cycle C_v must be within the region of a single controller.

If cycles C_s and C_t are in the same region then the controller for that region is responsible for at least $2QM > K$ (Inequality 2) flows then this would not be a valid partition.

Suppose an edge of the form $u^e v^e$ is in a cut of W. Then $M + 1$ flows will start at each of u^e and v^e.

We now show that I has a solution if and only if P has a solution.

Suppose there is a valid partition of W into $c = 2$ regions R_s and R_t. Then we know that C_s and C_t must be in different regions so without loss of generality we assume $C_s \subseteq R_s$ and $C_t \subseteq R_t$. If there are more than $n+1$ cycles $C_v \subseteq R_x$ for $x = s$ or $x = t$ then the controller of R_s is responsible for more than $MQ + M(n+1)D > K$ (Inequality 3) thus contradicting the assumption that the partition is valid. Therefore each of R_s and R_t contains exactly n cycles C_i other than C_s and C_t respectively. Thus the controller of each region is responsible for $M(Q + nD) + d$ flows where d is the number of arcs (u, v) where $u \in R_s$ and $v \in R_t$ (and similarly d is the number of arcs (v, u) with $v \in R_t$ and $u \in R_s$). Therefore if we assign $v \in V$ to V_s for $C_v \subseteq R_s$ and $v \in V$ to V_t for $C_v \in R_t$ this gives a equal sized partition of V into two equal sized subsets V_s and V_t with $s \in V_s$, $t \in V_t$ and d edges between them.

Suppose there is a partition V_s and V_t of V with $s \in V_s$, $t \in V_t$ and $|V_s| = |V_t| = |V|/2 = n$ where there are at most d edges crossing between V_s and V_t. Then define $R_s = \bigcup_{v \in V_s} C_v \bigcup C_s$ and $R_t = \bigcup_{v \in V_t} C_v \bigcup C_t$. Then it is easy to check that this gives a valid partitioning of W. ■

REFERENCES

[1] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete problems. pages 47–63. ACM, 1974.